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# "Students Misconceptions and Errors in Solving Algebra Word Problems Related to Misconceptions in the Field of Science"

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## ABSTRACT

This study relates domain specific misconceptions in mathematics to misconceptions in science. A set of propositional relation algebra word problems were constructed. These problems had the key contextual features of familiarity, imageability and variable type which interact with learner characteristics. The errors observed on these algebra word problems were due to the context of the algebra problems as opposed to the structure or content which are the source of scientific misconceptions and errors. Mathematical misconceptions, therefore, appear to be the result of naive cognitive operations which are epistomologically based.

## Introduction

For years misconceptions in mathematics, and particularly in algebra have been considered procedural or computational in character. In the last decade, however, a shift of attention from procedural knowledge to declarative knowledge has "promoted" what use to be known as "procedural errors" to misconceptions (i.e., conceptual errors). This shift, or reconceptualization, was due to the substantial research literature on misconceptions in the area of science: i.e., physics, chemistry, biology and/or earth science.

The general attribution of misconceptions in the areas of science and mathematics comes directly from intuitive notions about concepts that conflict with accepted scientific or mathematical theories. One characteristic of these misconceptions is their pervasiveness in all the disciplines. An example of this point is in the area of physics where students have difficulty describing physical phenomena in natural language and cannot interpret concepts from their equations or symbolic format (Kenealy, 1983). Similarly, in mathematics, students have difficulty translating algebra problems in the linguistic format to equations.

There are some commonalities between misconceptions (and errors) in science and mathematics. This study attempts to understand how particular features of algebra word problems, which copy real world representations, influence student responses and the errors they make and how these errors (i.e., misconceptions) are related to scientific misconceptions.

Little has been done to show similarities of misconceptions in mathematics and science. General misconceptions in mathematics tend to be known as "inconsistencies" in accepted mathematical theories and student's conceptions of these theories (Tirosh, Graeber and Wilson, 1990). Mathematics understanding tends to be essentially pedagogical. Abstraction or formal use of symbolic knowledge is rarely a matter of phenomenal experience, but of a set of well organized, structured axioms, rules and principles which are pedagogically driven. Thus, misconceptions in arithmetic (Tirosh and Graeber, 1990), algebra (Rosnick and Clement, 1980), graphs (Clement, 1989 and McDermott, Rosenquist, Popp and Van Zee, 1987) and functions (Vinner, 1983) give rise to the need of well grounded theory on misconceptions, as the theory has tended to be rambling and inconsistent in the literature. The main goal of this study is to provide a model of some aspects of these misconceptions and to relate mathematical misconceptions to misconceptions in the
field of science.

Generally, as stated misconceptions in mathematics, but particularly in algebra, graphs and functions have been essentially pedagogical in character resulting from "reasonable although unsuccessful attempts to adapt previously acquired knowledge to a new situation" (Matz, 1980, p.95). In other words, as evaluated by Tirosh, Graeber and Wilson (1990) misconceptions are represented as inconsistencies that arise between students' structure of mathematics and conventional mathematical theory. This "classical" view of misconceptions, is derived from the knowledge of science, or a domain area of science exhibiting the possibility of errors. These errors deal with phenomena brought from daily physical experiences; i.e., real life situation to a domain specific discipline or theory. This well illustrative point characterizes the anthropomorphism in physics, chemistry or mathematics which transcends itself to a functional processing of the operational approaches students' employ across several domain specific knowledge areas. For example, in physics, popular and known examples of these intuitive misconceptions are notions that only springs have the force to push back an object, whereas solid material have no equal and opposite force countering the weight of the object. Misconception in the perceived motion of an object given a force that occurs in this example leads to belief that that the object will move in the same direction as the force applied on that object (Di Sessa, 1982). There is no formal understanding that magnitude and direction are two independent concepts.

In mathematics, and particularly in algebra, welldocumented research on misconceptions has identified some robust misconceptions in the translation of word problems to algebraic equations. These errors are viewed as mistaken translations of nouns in algebra word problems to quantities and the mistakes are believed to occur due to the syntactical form of the verbal presentation (Clement, 1982; Sims-Knight and Kaput, 1983a and Gerlach, 1986). This erroneous approach to problem-solving has been constructed to be operational and labeled as the syntactical and a word-order match-naive approach. A student does not immediatly see the quantitative relation in the problem, but instead operates by decoding the problem in a syntactic manner. The general view of these errors is so rigidly entrenched, that it is ferverently held that these errors arise from instruction, either formal or informal. However, in mathematics there exists some subtle and intricate errors students make that are analogous to scientific misconceptions. These errors , moreover, are attributable and to some of the causes of scientific misconceptions.

We will attempt to identify below some of the domain specific errors in algebra, which are similar to those found in science by analyzing errors in the context of carefully constructed algebra word problems. Our view is that these errors are transcendent and cross all content domains. Furthermore, these errors are essentially intuitive and have structural characteristics similar to misconceptions found in scientific knowledge (Posner, 1982).

We have constructed and validated a domain-referenced set of algebra word problems (see Nasser and Carifio, 1993a). These problems have systematically varied key contextual features. Contextual features information is important nonstructure, non-operational information that might help students process and meaningfully generate a solution to the problem (Caldwell, 1984). Contextual features are the "clothing" of the generic structure of the algebra problem.

Algebra word problems have been widely used to study a specific error called the reversal error (see Rosnick and Clement, 1980; Clement, 1982; Lochhead and Mestre, 1988; Mestre and Gerace, 1986 and Niaz, 1989). The uniqueness of the present study is in its method of first identifying the type of error produced on the propositional relation problems and then establishing the nature of the relation between the errors and the features of the algebra word problems. In this way, the experiential phenomena which may be attributed to the algebra problem can be related to the error, and consequently certain errors that are projected from the student's conceived world will be identified with the phenomena of the real world. Thus, one may to some degree understand the influences of real world phenomena on student's approaches and responses to the algebra word problem.

We constructed a domain referenced set of 16 algebra problems. All of the 16 problems are of the propositional relation kind. Mayer (1981, 1982) described these type of problems as being propositional because a relation is established between two variables in the problem. Students determine equivalences by figuring out the proportions for the variables in the problems.

Our problems have these different formats. These three formats are the pictorial, symbolic and verbal modes of problem presentation and problem response form. The problems have key contextual features assigned and varied on a continuum from a concrete feature (i.e., familiar or readily imageable) to more abstract. The contextual domain here is considered as being information and not connected to the problem's structure or the mathematical operations needed to solve the problem.

This systematic variation in presentation, response, and context variables should help us to understand how experiential reality might be influencing students problemsolving solutions. We should, for example, be able to infer if certain features are invoking shared schemata, which correspond to some aspects of the real world phenomena, and which are similar to the problem presentation or context variables, and if so, then the student should be able to approach and solve the problem. For instance, a student is given a problem about vehicles. The context and proposition of the problem is "for twenty wheels there is one car." The student is then asked to set up an equation.

Preconceptions could arouse an answer to a real and conceived fact about cars (that they have four wheels), which in turn leads the student to erroneously provide the wrong solution. In some other situations, the presentation form, such as the pictorial format, may cause students to set up a mathematical equivalence that is epistemologically representative of the phenomena because knowledge per se according to many epistemologists originates in experience

(Strike and Posner, 1985). This type of misconception tends to appear when the presentation format is pictorial, formal, linguistical that has a specific symbolic system.

Clement (1989) suggested that a pictorial presentation of the concept of variation is often seen as a static model and an analogic model to a symbolic represention. One of Clement's discriptions of a real-life representations is a "figurative correspondence between the shape of the graph [or pictorial presentation] and some visual characteristics of the problem scene" (Clement, 1989, p.82). These errors in a domain specific area of mathematics are analogous to specific examples in science. For example, in the area of physics, students erroneously predict the direction of a trajectory of a moving sphere (McCloskey, 1983). Small children believe that the earth is flat (Nussbaum, 1979) which is a graphic illustration of the sort of thing that learners do that fits their real knowledge about the physical world (Pines and West, 1983), as they do in some cases in mathematics.

<u>Purpose</u>

We sought to assess the type of errors students make in solving algebra word problems in terms of specific key context features and problem presentation and responding formats. We believe that analysis of context and presentation features should reveal the phenomena associated with the error students make, which in turn could help to answer the main question which is: "Does experience of physical phenomena have an influence on students erroneous responses on the algebra problems." These attributive misconceptions are analogous to what is called naive approaches or intuitive misconceptions in the literature on scientific knowledge. We will attempt to establish the commonalties between some of the errors in these two domains; namely mathematics and science.

<u>Methodology</u>

We initially constructed twenty word problems. These problems were later reduced to 16. The twenty word problems constructed had three different presentation formats. These three presentation modes were of the pictorial, verbal and symbolic modes. Each mode of presentation of a problem had three modes of answering the problem, namely, a pictorial, verbal, or symbolic response format. Therefore, students had to process and translate each problem from its presentation mode (pictorial, verbal or symbolic) into a particular response format; namely, pictorial, verbal, or symbolic outcomes or answers. Consequently six "cross-translation" combinations (or modes) were possible (see Table 1 for details).

The key features of familiarity, imageability, and variable type (discrete and continuous) were the main constructs of interest in these 16 algebra word problems, as these attributes individually have been shown to effect performance on arithmetic and algebra problems (e.g., SimsKnight and Kaput, 1983a; 1983b; Lyda and Franzen, 1945; Sutherland, 1942; Brownell and Stretch, 1931; Washbrone and Osborne, 1926 and Horwitz, 1980). It should be noted no study of algebra word problems have employed more than one of these key contextual features to study their effects on the problem-solving processes. When these three modes of presentation and their cross translation are combined with the key features of familiarity, imageability, and variable type, one gets the domain of possible algebra word problem type described in Table 1.

As can be seen from Table 1, all of the verbally presented problems were given triads of attributes, following the continuum expressed in concretness to being more abstract in four levels: (1) familiar-readily imageable-discrete; (2) familiar-not readily imageable-continuous; (3) unfamiliarreadily imageable-discrete and (4) unfamiliar-not readily imageable-continuous. The theoretical rationale behind this nested design and the operational definitions used for these features is given in Nasser and Carifio (1993b).

The verbally presented problems were created to have the above features. However, these very features were very hard to assign to the pictorial and symbolic presented problems. Hence, the symbolic and pictorial problems in this

Table 1: <u>A Descriptive and Conceptual Characterization of the Domain of Algebra Word Problem.</u>

| Mode of Representation and Cross Translation | Key Contextual Features | | | |
|---|---|---|---|---|
| | FI/D | UI/D | FU/C | UU/C |
| Verbal to Symbolic | 1 | 1 | 1 | 1 |
| Symbolic to Verbal | | 1 | | 1 |
| Pictorial to Symbolic | | 1 | | 1 |
| Symbolic to Pictorial | | 1 | | 1 |
| Verbal to Pictorial | 1 | 1 | 1 | 1 |
| Pictorial to Verbal | | 1 | | 1 |

FI/D= familiar-readily imageable-discrete
UI/D= unfamiliar-readily imageable-discrete
FU/C= familiar-not readily imageable-continuous
UU/C= unfamiliar-not readily imageable-continuous

Note: The acronym FI/D is interchangeable with FI, similarly FU/C with FU, UI/D with UI, and UU/C with UU

study were limited to the following attributes of unfamiliarreadily imageable-discrete quantities (UI) and unfamiliar-not readily imageable-continuous (UU). The reason we used triads and a nested design essentially boils down to the single fact that a group of students can only do so many algebra word problems in one setting and, therefore, one must chose the most relevant feature combinations for the problems to study initially.

Many researchers have viewed student difficulties in solving algebra word problems, especially of the propositional kind, as basically a problem in student's handling of the verbal structure of the problem (e.g., Mestre, Gerace and Lochhead, 1982; and Mestre and Gerace, 1986). This framework, however, represents a very limited view of algebra word problems and problem solving behavior. No researcher in this area has approached algebra word problem in terms of the various modes in which the problem may be presented (i.e., pictorially, verbally or symbolically), or the mode of representation of the answer to the problem, nor in terms of the translation of the relations in these various modes of presentations from one mode to another. Further, no research has categorized or studied the errors students make in terms of presentation, responding, and cross-translation formats. These very basic limitation in the research literature, is one of the driving force behind this study.

Validity and Reliability

Validation of the 16 problems was done in two phases: the first, a consensual validity phase, and the second, a construct validity and reliability phase (see Nasser and Carifio, 1993a for details). The consensual validity study assessed the problems for their adequacy, quality and appropriateness. Because the key context features of the problems are the operationalized constructs, it was necessary to assess the actual presence of the stated features in each problem (i.e., their construct validity).

The method used to assess the construct validity of the 16 word problems was an adaptation of Campbell and Fiske's (1959) convergent and discriminant validity paradigm. In general, this method compares independent ratings of an operationalized construct with that of an expert (or correct answer key). The strength of this model derives from the addition of raters. If two or more judges agreed on the operationalized construct, this would be evidence to the items logical validity; namely that the item reflected what it purported or claimed to reflect (Dagostino and Carifio, 1993).

The main constructs of this study were each problem's key context features and variable type. These problems were validated by six mathematics educators. Each judge was given the 16 problems in a different random order, the 6 judges had to read each problem and indicate which key features were present in the problem. Judges were given operational definitions of the features and one training session with examples.

Overall, the results were very positive; only 6.5% of the ratings of the key features were incorrect. For the 16 algebra problems, the raters, therefore, agreed with each other and were correct on 93.5% of the ratings. All six judges rated 8 problems correctly. Of the 9 remaining problems, there were three or less raters who disagreed on the key contextual features present in the problem. When one item which had a high incorrect response judgement among raters was removed from the analysis, no statistical significance difference was found among raters, across all items. The results indicates both strong reliability and validity for the 16 algebra problems.

For the six raters, interrater reliability score among 51 ratings for all the attributes was at R=+.95. Haggard's (1958) ANOVA procedure was used in computing the interclass R. The interrater reliability for familiarity, imageability and variable type quantities classifications was at +.93, +.95 and +.97 respectively. As expected, the lowest interrater reliability was observed on the familiarity classification because of a scattered ratings in the profile between different raters within different items. Coefficients indicates that both correctness and agreement levels were extremely high.

Types of Errors

The errors examined in this study were derived from the existing literature. A majority of these studies attempted to explore errors as they related to the linguistic structure of the verbal problem students attempt to solve Clement (1982); Clement, Lochhead and Monk (1981); Wollman (1983); Rosnick and Clement (1980); Gerlach, (1986); Niaz (1989) and Mestre (1989). The majority of these researchers found that students (for the most part first-year engineering students and/or students who have taken two semesters of calculus) committed a "variable reversal error." For example, given the propositional relation problem:

> Write an equation, using the variables S and P to represent the following statement: There are six times as many students as professors at this university. Use S for the number of students and P for the number of professors.

Students usually reversed the coefficients on the equations by writing, "6S=P", instead of writing the correct response to the equation, "6P=S." A number of the studies mentioned above have viewed the reversal error in terms of a cognitiveoperations model in which students are consciously aware of the quantities, but in their attempts to device a solution, they make a translation error (see Rosnick and Clement 1980 and Clement, 1982). Several other studies have associated reversal error with linguistic factors (e.g., Clement, Lochhead and Monk 1981; and Gerace and Clement, 1986). However, none have attempted to evaluate the stimulus and to view the error or type of errors in terms of a conceptual framework.

Two error types were derived from the literature. These two were the reversal error and qualitative errors. Other types of errors considered were responses that had no perceivable logical relationship to the problem. Such responses are termed as incongruent responses. In our methodology and mode of analysis, the approach is to find a pattern in which the presentation mode or key feature can be attributed to a type of error. Our scoring scheme (see Table 2) for the algebra word problems allowed us to analyze the errors made by students quantitatively without further qualitative evaluations.

Table 2: Scoring Code for the Passive and Generative Translations.

| | Error | Correct Response |
|---|---|---|
| 1. | Incongruent Response | |
| 2. | Reversal Error | |
| 3. | No Response | |
| 4. | Qualitative Response | |
| 5. | | Static Correct Response |
| 6. | | Correct Response |

1. An incongruent response is an error which has no reference to the question;
2. Reversal errors are mismatches of the variables to their proper coefficients;
3. for no solution are unanswered problems;
4. A qualitative response is a solution in which the respondent presents the answer in qualitative terms, giving the greater or less than relation but not specifically the correct answer;
5. A static correspondence is given for the formulation of an equation, that does not express a proportion but to the quantities represented. This score was formulated after piloting the test;
6. Students formulate an answer to the problem

## Sample

A convenience sample of 80 students from a large commuter university in the eastern part of the United States was used. Only 37 college students completed the two testing periods. All college students were 19 years of age and above.

A second convenience sample of 193 secondary school students was obtained from two large high schools which serve two cities in the eastern part of the United States. Each of the two cities had a population greater than 60,000 people.

The age of the composite sample ranged from 11 to 40. The mean age was at 17.25 and median age at 17.00. All the high school and college students had successfully passed their first and second algebra courses. Two periods were allocated for administering the 16 algebra word problems we devised. One problem was added to validate the familiarreadily imageable features of the algebra problems.

## Results

The analyses of errors performed were done by isolating the features of the problem (i.e., imageability, familiarity, variable type and the presentation mode). These isolated features work as "blocks" when crossed with the type of errors. In this way, the most frequently occurring error can be identified for the type of feature or presentation type. We hypothesized from cognitive learning theory that those features that were the closest analogs of reality would be misprocessed most frequently by naive students who tend to base their processing on their experience with the real world, rather than the actual feature of the problem.

For example, in terms of the propositional relation problem, students may exhibit a clear understanding of the notational system and be able to use and manipulate the symbols required in the algebra problem. However, they may have little awareness or understanding of the "deep" dimension of the problem structure and/or underlying tacit knowledge needed to solve the problem and the errors they make, therefore, this study will reflect these deeper levels or dimensions. Thus, the errors derived from the literature and given in Table 2 are taxonomical and may be thought of as "degrees of misconceptions" when related to the feature or presentation mode of the algebra problem. At the lowest level are the incongruent responses followed by the reversal error, no response and lastly the qualitative response.

To illustrate our taxonomy for the reader, students are given a problem whose presentation is symbolic. Logically, this type of problem would have an abstract and irregular features relative to the experiential world. Hypothetically, then this type of problem should tend to produce either an incongruent response or a reversal error because of its irregular and abstract presentation. These errors are at the front end of processing in attention, perception and working memory. A pictorial presentation whose perceptual schema may be relevant to a previously stored piece of information, on the other hand, is essentially concrete and relies on a previously assimilated piece of information. Consequently, pictorial presentation should produce qualitative errors most frequently as appropriate information is "paged" from long term memory to working memory, but cannot be appropriately converted to a full and correct solution. Qualitative errors, therefore, occur late and "deep" in the information processing cycle. Furthermore, qualitative errors are partially, but not fully correct, and they are due to something other than the objective character of the problem or the students experiences.

The first analysis done was on the key contextual features of familiar-readily imageable (FI) and familiar-not readily imageable (FU) for the verbal to symbolic (VS) and verbal to pictorial (VP) cross translation problems. Table 3 presents the frequencies of errors for the verbal presentations of the key contextual features of FI and FU features. Table 3 was constructed by combining the VP modes and the VS modes in terms to FI and FU assignments. A significant Chi-square statistic was obtained for Table 3 ($X2=196.1$, d.f.=3, $p<.05$). There was a higher proportion of errors on the VS and VP problems with the familiar-not readily imageable (FU) assignments than the VS and VP problems with FI assignments. These results indicate that reversal errors in algebra problems presented in the verbal form and translated to a symbolic or pictorial mode were highest with those problems having key contextual features of the familiar-readily imageable type assignments. The largest number of qualitative responses were found on those problems which were of the familiar-readily imageable type, whereas, the largest number of unanswered prolems was indicated on those problems of the familiar-not readily imageable features.

Table 3: Count of Errors for the Subset of Problems of the FI and FU Key Contextual Features.

| | Incongruent Response | Reversal Error | No Answer | Qualitative Response | |
|---|---|---|---|---|---|
| VS&VP/FI k=2 | 49 | 161 | 40 | 12 | 262 |
| VS&VP/FU k=2 | 101 | 29 | 117 | 84 | 331 |
| Total= | 150 | 190 | 157 | 96 | |

VS&VP/FI= verbal to symbolic & verbal to pictorial problems with familiar-readily imageable features
VS&VP/FU= verbal to symbolic & verbal to pictorial problems with familiar-not readily imageable features
k=number of problems

Table 4 presents the frequency of errors on the unfamiliar-imgeable (UI) and unfamiliar-not readily imageable (UU) type of problems. A total of 905 errors for the problems of the UI features were found and 773 for the UU features yielding a significant Chi-square of 93.56, d.f.=3, $p<.05$. As can be seen from Table 4, these results indicate a larger number of errors made on problems that were readily imageable than the not readily imageable problems with unfamiliar key context feature.

Other findings that may be seen in Table 4 were a high number of errors found on those problems of the UU type on all the problem presentations than on the UI problem type. For those problems that have unfamiliar features, a large number of errors were found on the not readily imageable features versus those that were readily imageable, whereas, for the familiar features a larger number of errors were found on the not readily imageable features versus those that were imageable ones.

Given the results presented in Table 4, it is clear that the key contextual features and problem presentation mode are operating together to influence students responses on the propositional relation problem. This point is evident from the number of errors resulting from the verbal problems with FI, FU, UI and UU key contextual features as compared to the pictorial, symbolic and verbal presentation with the UI and UU features. The latter had a large number of errors on the familiar-readily imageable assignments (FI), versus the unfamiliar-not readily imageable assignments (FU), as compared to the former, which had a large number of errors on the unfamiliar-not readily imageable (UU) versus those unfamiliar-readily imageable (UI). Thus, when combining all the problem presentation modes together in terms of the UI and UU features, the largest number of errors were on the UU features. When combining all the verbal problems based on the FI, FU, UI & UU feature, the largest number of errors were found on the FU features, underscoring the point that the problem presentation modes and key contextual features are interacting together to influence the responses (i.e., error mode).

Table 4: Frequencies of Errors on All Problem Translations Combined in Terms of all the UI and UU Key Contextual Feature.

| | Incongruent Response | Reversal Error | No Answer | Qualitative Response | |
|---|---|---|---|---|---|
| UI | 210 | 505 | 121 | 69 | 905 |
| k=6 | | | | | |
| UU | 187 | 271 | 190 | 125 | 773 |
| k=6 | | | | | |
| Total | 397 | 776 | 311 | 194 | |

Tables 5 and 6 presents an average count of errors for the three presentation modes and their cross translations respectively. The average was used because there was an unequal number of problems for the presentation and cross translations modes. The findings in Tables 5 and 6 reveal that the reversal errors were persistent among the presentation to and from the symbolic form of presentation. The results showed a high average of 86 reversal errors on the translation from a symbolic mode compared with an average of 75.17 on a translation to a symbolic mode. For the verbal formats a high reversal error was found on the translations from a verbal format, with a comparable average on the translations to a verbal format. The high occurrence of the reversal error on the symbolic presented problem and its cross translation to a symbolic form follows from what Clement (1982) called the syntactic order approach, in which students operationally decode the presentation syntactically as it appears in its linguistic form.

The average number of qualitative response errors was relatively comparable among the three translations from the pictorial, symbolic and verbal format. The highest average counts of qualitative responses was found on the pictorial presented problem. In contrast, the cross translations had a higher number of qualitative response errors. On the translations to a verbal format, this high number of qualitative response errors to the verbal mode gives strong validity that these translations were made from the pictorial format. In summary, the results above indicate that those presentation formats that are qualitative (and thus generative) in the character they appear to have the highest number qualitative response errors.

Table 5: Average Number of Errors From a Translation.

| Translation Representation | Incongruent Resp. | Reversal Error | No Resp. | Qualit. Resp. |
|---|---|---|---|---|
| From Pictorial k=4 | 34.5 | 42 | 22.5 | 26.3 |
| From Symbolic k=4 | 27 | 86 | 8.75 | 19.25 |
| From Verbal k=8 | 37.63 | 56.75 | 43.86 | 20.5 |

Table 6: Average Number of Errors by a Cross Translation to a Response Mode.

| Translation to a Representation | Incongruent Response | Reversal Error | No Resp. | Qualit. Resp. |
|---|---|---|---|---|
| To Pictorial k=6 | 30 | 00.5 | 39.33 | 17.67 |
| To Symbolic k=6 | 20.17 | 75.17 | 24 | 09.67 |
| To Verbal k=4 | 13.5 | 59 | 04.5 | 32.25 |

Highest count of unanswered problems was on the translation from the verbal format, with an average count difference between verbal and symbolic presentations of 35.11 and a difference of 21.36 between the verbal and pictorial represented formats. As expected, those problems that were more complex perceptually had the highest number of unanswered problems. These problems were verbal presentation problems as well as their translations to the symbolic format and to the pictorial format. The average frequency data suggests that the large number of unanswered problems were mostly of the generative problem type.

In summary, the highest average number of reversal errors was found on the symbolically presented problems and the symbolic cross translation. These averages show that symbolic presentations and cross translations may have encouraged the solver to translate these problems in a syntactical manner. This finding is what Clement (1982) called the syntactic word order match operation or the "adjacency effect," where students operationally match the quantity to the object as it syntactically appears in the form of presentation; i.e., the naive processor approach.

## Discussion

The main purpose of this study was to understand how the key contextual features in algebra problems are related to the errors students made in solving the problems. Students usually hold intuitive misconceptions or preconceptions about the problems that conflict with the theoretical-formal rules of a mathematical theory.

Several algebra word problems were constructed to understand the effects of the presentation and key contextual features on students performance on a set of a propositional relation problem type (Mayer, 1981; 1982). These problems were presented in pictorial, symbolic and verbal forms and had the key contextual attributes of familiarity and imageability, and the variable type (discrete and continuous variables). When these features were crossed with the problem presentation and responding formats, 16 domain referenced problems resulted (see Table 1).

The 16 problems had two responding formats which required a generative or a passive translation of the problem into an answer. In the generative formats, the student has to construct the correct answer, whereas in the passive format, the student had only to select the correct answer. These type of response formats or "translations" were in accordance with Clarkson (1978) view of the cross translation from a presentation to a response mode.

Logical analysis revealed that in some definite cases the translations are convergent (VS, PS and VP translations) and in others, they are divergent (PV, SP and SV). Hence, the domain referenced set of problems were reduced to 16 problems.

The first analyses performed on the verbal problems was done by combining the verbal and pictorial problems to analyze the effects of key context features of the FI and FU type. The results showed a larger number of errors on the familiar-not readily imageable (FU) type versus the familiarreadily imageable (FI) ones. A large frequency of reversal errors were found on those problems that were of the FI features versus those of the FU. These results showed that students who use naive approaches are foiled by certain key features which interfere with task conception and problemsolving processes. These naive students tend to view contextual features as part of the problem's structure.

For example, in a verbal problem with the familiar-readily imageable (FI) key context feature that relates the number of nickels to the number of dimes, the student is asked to translate the problem's structure from the verbal mode, to the symbolic mode (i.e., equation). Most of the students take at face value the currency value of the coins (i.e., a dime is 10 cents and a nickel is 5 cents) versus the relation between the number of nickels and number of dimes as stated in the problem. Hashweh (1986) found that this same type of erroneous approach is pervasive in children's learning of specific scientific theories (i.e., scientific knowledge). "Children learn the quantity of liquid in a glass is affected by the height of the glass" (Hashweh, 1986, p. 234). They later discover in addition to the latter influences is the area of the container.

Children tend to use this conception in other areas of scientific knowledge; e.g., children have the notion that the "mass of the pendulum bob effects its period (Hashweh, 1986, p. 234), which in their conception is unrelated to the size of the bob. The familiarity and imageability key context features of the verbally presented problems, work analogously to certain instantiations found in scientific knowledge when schematized to some aspect of the real world or phenomenalogical experiences (i.e., the experiential world). Many naive approaches to solving algebra word problems can be mapped to aspects of the features of the experiential world and the relations between these aspects that form certain macro-aspects which pervade student conceptions and are detrimental to student performance on algebra problems.

Given that we are drawing parallels between scientific misconceptions and mathematical ones, the analogies may not be truly and completely aligned: Scientific misconceptions, as found in the literature in science education (e.g., Driver and Easely, 1978; Hashweh, 1986; Kenealy, 1983 and more recently Vosniadou and Brewer, 1992), are content based and drawn from the experiential reality of each child. When mathematical content is involved, however, the majority of misconceptions tend to be related to interaction between instruction and the learner; i.e., errors in mathematics

become pedagogically driven in the intrinsic ways instruction interacts with the language of mathematics and
students inability to assimilate and accomodate mathematical ideas, meanings and systems (i.e., cognitive dissonance and interference). It is well established that symbolic knowledge is organized and restructured in the conceptual organization of the language we learn as children. The main premise, and at the heart of the matter, is that intuitive misconceptions (or naive theories, preconceptions or folk theories) in mathematical knowledge are all products of context rather than structure (which is called content in science). Since the variables of context, content and structure of the mathematical problems are interrelated and cannot be easily separated to study the effects of one variable over the other, analogies drawn from the effects of context in mathematical problem-solving to the effects of and content in scientific knowledge are some what circumstantial and should be viewed with great caution until an underlying theoretical connection between the two is made as we have attempted to do here.

The high frequency of qualitative and no responses on the verbally presented problems seemed to be encouraged by the not readily imageable feature, which seems to hinder students from evaluating these problems operationally. Students tended to shy away from those problems that were not familiar to their formal instruction or "worldly" experiences. When students attempted these problems operationally, they were discouraged from setting up an equivalence, and as a result produced the formulae qualitatively, giving the relation in greater and lesser quantities. These results were also found with all the problem presentation modes combined, based on the contextual features of unfamiliar-not readily imageable (UU) versus unfamiliar-readily imageable (UI).

The qualitative responses were found in high frequencies on the UU problems that had not readily imageable features of the problems. The not readily imageable features may be operating as "devices" that alert students to the fact that they are faced with a real and non-routine problem which in turn helps students to approach and solve the problem at least qualitatively by giving an answer in greater and lesser terms. Therefore, qualitative response errors on the verbal problems resulted from key contextual features that were not depictable or readily available in students' knowledge structures. Unanswered problems also seemed to be encouraged by the not readily imageable features for both features of the familiar and unfamiliar features. A general interpretation of these results would be that students are more prone to answer problems related to their phenomenal experience or to some schematic structure, formal instructional and familiar to them.

As stated earlier, two major types of misconceptions seem to characterize misconceptions in science. The first is the naive approaches used by naive thinkers; namely, students who tend to have and maintain naive views of the problem presentation as generated from real world objects (Larkin, 1983). Larkin states that naive approaches are naive representation for envisionment and are "internal representation of the problem that contain direct representations of the familiar, visible entities mentioned in the problem, stimulating the interaction of these entities through operators, [to predict] subsequent events on the basis of former events ... (Larkin, 1983, p.77)." Naive representations in students' are direct result of real and imageable objects according to Larkin (1983). For Larkin and a great many others, the problem is basically a representational one, which interacts with students intuition.

Most researchers agree that intuitive misconceptions can come about from formal instruction. Typically, in science, instantiations of these type of misconceptions are found in students' understanding of theories of motion (Mcloskey, 1983) in that setting an object in motion entails it has an internal force or so called impetus force. When the object comes to rest it is thought that the impetus force dissipates from the object. This type of conception has historical roots, and Clement's (1983) review of 17th century work by Galileo's revealed similar type of misconceptions held by Galileo about physics concepts. Thus, intuitive misconceptions may be very subtle and deeply rooted making them far more difficult to remediate as compared to the context and procedural based misconceptions and errors, typically found in mathematics.

The development of physics into a modern day science from the time of Galileo is in some way analogous to the (probable) development of cognitive reasoning and the conception of real world phenomena, as well as the development of more formal notions of the theory of the phenomena selected for this study. A student's naive representation of a physical phenomena is abstracted through a set of operations and processes (i.e., attention and perception) that consequently define the next set of events needed for the problem-solving situation. Intuitive misconceptions are errors that ripple and multiply through the whole problem-solving process.

The second major type of misconception in science is the instructional misconception. This type of error is explored by Matz (1980) and is what Hashweh (1986) calls the "set effect" in which conceptions formed in certain situation are extrapolated to a new one whose representational characteristics seem to be similar to the old problem situation but are not. This lack of appropriate discrimination, and inappropriate generalization, occurs a great deal in a wide variety of instructional processes. This subtle error, which may occur during the process of eliciting deep seated procedural knowledge, is not easy to extinguish even with intervention and remediation (Clement, 1982).

One of the more important aspects of this error may be observed in our algebra problems. Our problems may be aggregated to view the errors in terms of problem presentation and its response modes. The significance of this method of aggregation is that it exposes more clearly the type of misconception (i.e., instructional or naive view) with which the errors are associated. This particular analysis found that the reversal error were highest among those problem presentations of the symbolic form and cross translation to symbolic form. These presentations and the cross translations have a syntactical format similar to the modes of natural language sentences syntax. This particular syntactical format facilitates the reversal error.

The verbal format tend to have static correspondences of word-nouns to their adjectives producing the "adjacency effect" in this representational format. When students naively approach any of the isomorphic pictorial, symbolic and verbal propositional relation problems, the tendency is to view these problems (naively) as ordered in a patterned and syntactical way, which is used wrongly to guide the writing of mathematical relations in an equivalent syntactical response mode. Thus, these problems when decoded produce misconceived notions of theories describing the physical phenomena; e.g., predicting the trajectory of a
moving sphere in scientific knowledge (Mcloskey, 1983). Syntactical patterns in representations do not necessarily correspond with structure, and naively or unconsciously assuming that they do is both the misconception and deep misunderstanding that is operating. Further, it is in this key insight that we catch a glimpse of an important point about misconceptions which is that they often reflect a lack of metacognition and metaknowledge on the part of the student or misconception holder. We call these errors, metacognitive misconceptions.

The qualitative response errors were logically viewed by us to have qualitative formats; i.e., pictorial presentations. Qualitative responses were found in large numbers especially on those problems which were of the pictorial response format. Qualitative responses were also high on problems in the verbal format. These findings suggest that those presentations which seem to have perceptually complex features have a large number of qualitative responses. The more complex the presentation the harder it is for students to naively approach the problem.

In summary, our work on errors students make when attempting to solve propositional relation algebra word problems establish some facts about misconceptions that are related to the problem's features. One of our important findings is the similarities in students' misconceptions and errors on algebra word problems and those that have been found in the area of science. Next is our finding that students' naive approaches or deep seated preconceptions have two main effects on the process of solving an algebra word problem. The first is that these preconceptions interfere with problem-solving processes when the context is similar to real life events described in a different way. The problem, therefore, is not representational problem, but rather a re-representational problem which is a much more difficult problem.

The second finding is that misconceptions conflict with a more appropriate conceptions of real life events, and in this conflict, in misconceptions are the stronger of the two "combatants" (Hashweh, 1986). In a word, appropriate conceptions do not easily or automatically root out misconceptions, and this golden premise of the enlightment is incorrect. Thus, the key contextual features of familiarity and imageability may "lull" students into a false sense of control, and they may be less vigilant and attentive in their work, which they may not double check for correctness because the problem does not seem to be a real problem. Problems whose features are unfamiliar and not imageable, on the other hand, may act as "screaming demons" which trigger high vigilance and attentiveness behavior and the double checking of work, because it becomes quickly apparent from the unfamiliarity and unimaginable features of the problem that one is confronted with a problem that just might not be relatively routine and "a piece of cake."

One way to counter the naive approaches in mathematics is to encourage students to approach these problems scientifically (Perkins and Simmons, 1988). Epistimological frames could be employed to understand initial conceptions as they related to new ones. However, this approach might not be easily carried out as it may require students to operate at an increased formal levels of symbolic reasoning. students need to develop and acquire the knowledge to analyze and break the problem down. Consequently, we are currently studying Piagetian formal reasoning levels as predictor of the type of errors we have observed which may help us in understanding the underlying structure of our error taxonomy.

In conclusion, it is clear that the reversal errors and qualitative response errors result from naive and proceduralized "equation cranking procedures," which are affected and short-circuited by the surface features of the problems. The general pattern of errors, (i.e., the reversal errors, qualitative errors and unanswered responses) were affected by a "surface" dimension of key contextual features. A "deep" dimension, which may be attributive to the structure of the algebra problem, seems to be interacting with the surface dimension to produce specific types of errors. In a word, naive misconceptions are very resistant to change and resurface when there is no awareness of the deep and underlying structure of the knowledge domain in question i.e., structure of the problem one is trying to solve. And structure is first, foremost, and lastly a perception, conception and higher order cognitive act and phenomenon, which is a simple fact that we all need to keep constantly in mind. Metacognitive misconceptions, therefore, need to be more fully investigated if we are truly to understand, remediate, and alleviate the problems of misconceptions in the sciences and mathematics.

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