# Third Misconceptions Seminar Proceedings (1993)

Paper Title: **Transitional Conceptions: An Alternative Perspective of Students' Conceptions and Their Role in Instruction**

Author: Moschkovich, Judit

Abstract: Research on student conceptions in mathematics has documented particular student ideas and described how they are at variance with expert ideas. However, it has neither presented a comprehensive account of student conceptions nor resolved crucial questions regarding the nature and transformation of these conceptions (Smith, diSessa, and Roschelle; in press). Analyses of student conceptions describing errors and misconceptions have focused largely on the "mis-" aspect of student ideas and have not considered conceptions that may be useful, applicable in some contexts, or productive for advancement. On the other hand, while the term "alternative interpretations" shows a certain respect for student ideas, its use misses the point that while there may be many alternative ways to conceive of a domain, there is a mathematically accepted way to think about the subject matter.

Keywords:
General School Subject:
Specific School Subject:
Students:

Macintosh File Name: Moschkovich - Mathematics
Release Date: 12-16-1993 C, 11-6-1994 I

Publisher: Misconceptions Trust
Publisher Location: Ithaca, NY
Volume Name: The Proceedings of the Third International Seminar on Misconceptions and Educational Strategies in Science and Mathematics
Publication Year: 1993
Conference Date: August 1-4, 1993
Contact Information (correct as of 12-23-2010):
Web: www.mlrg.org
Email: info@mlrg.org

A Correct Reference Format: Author, Paper Title in The Proceedings of the Third International Seminar on Misconceptions and Educational Strategies in Science and Mathematics, Misconceptions Trust: Ithaca, NY (1993).

National Science Foundation, and the transformation of this collection into a modern format was supported by the Novak-Golton Fund, which is administered by the Department of Education at Cornell University. If you have found this collection to be of value in your work, consider supporting our ability to support you by purchasing a subscription to the collection or joining the Meaningful Learning Research Group.
----- -----

# Transitional Conceptions: An Alternative Perspective of Students' Conceptions and Their Role in Instruction

Judit Moschkovich
EMST/SESAME
University of California
Berkeley, CA 94720 U.S.A.

**Introduction**

Research on student conceptions in mathematics has documented particular student ideas and described how they are at variance with expert ideas. However, it has neither presented a comprehensive account of student conceptions nor resolved crucial questions regarding the nature and transformation of these conceptions (Smith, diSessa, and Roschelle; in press). Analyses of student conceptions describing errors and misconceptions have focused largely on the "mis-" aspect of student ideas and have not considered conceptions that may be useful, applicable in some contexts, or productive for advancement. On the other hand, while the term "alternative interpretations" shows a certain respect for student ideas, its use misses the point that while there may be many alternative ways to conceive of a domain, there is a mathematically accepted way to think about the subject matter.

The problem is not only in the theoretical perspective, but also in what types of student responses or ideas are the objects of analysis. The analysis of errors or bugs emphasizes the procedural aspects of a domain, does not address differences in terms of conceptual content, and usually refers to responses on limited tasks. Some student responses are mistakes or errors which can be addressed directly and erased or replaced by the correct convention (for example a reversal of the coordinates of a point). Some errors are the result of repairs to achieve an answer on a written problem, while others can be related to deeper conceptions or principles. Other student conceptions may, at first glance, look like simple mistakes or misconceptions but may in effect be useful conceptions which are sometimes applicable and have the potential to be refined.

Understanding learning in a complex domain such as linear functions necessitates a perspective which lies somewhere between the two extremes of seeing student conceptions as either wrong or simply alternative. In the following discussion I will attempt to balance the positive and negative aspects of student conceptions, while still acknowledging the tension between mathematically accepted conceptions and those conceptions students generate. I will introduce and use the concept of "transitional conception" as a way to include important conceptual knowledge which is not always an error, which is sometimes useful (depending on the context), and which can change by refinement.

While initial conceptions are sensible to the learners themselves, they are also different from expert conceptions; different objects may seem relevant, the relationships among objects may be unspecified, and language usage may be different and ambiguous. Students' use of the x-intercept for equations of the form y=mx+b will be described in terms of the objects and relationships that students see as relevant. While the refinement and transformation of student conceptions will not be discussed in this paper, I assume that these occur through inherently social processes which are mediated through language[1].

In a previous paper (Moschkovich, 1990) I summarized students' interpretations of linear equations and their graphs documented in two algebra classrooms. Written assessments designed to explore these interpretations in greater depth were completed by 18 students from these classrooms. The responses on these pre-tests showed that 13 of the students in this study (72%) used the x-intercept at least once when working with equations of the form y=mx+b, in place of either the parameter b (12 students) or the parameter m (5 students) in their equations. Six of the students also described lines as moving left to right (or right to left) along the x-axis as a result of changing b in an equation. This conception was also evident during the discussion sessions and on the post-test responses of several students[2] .

These uses of the x-intercept could be considered as misconceptions, errors, or even a simple mismatch with the convention that the x-intercept does not appear in equations of this form. On the other hand, the use of the x-intercept could be understood as a reasonable conception which reflects the mathematical complexity of this domain, which is applicable in some contexts, and which can be refined. In this paper I will summarize the analysis of the videotaped discussion sessions for two pairs of students to support the claim that the use of the x-intercept is an example of a transitional conception rather than an error or a misconception.

The data from the discussion sessions show that the use of the x-intercept was more than a superficial mistake from which students could easily recuperate. This conception became a central point of the students' exploration and discussions. Most of the students spent a considerable amount of time discussing the x-intercept and making sense of their responses regarding the x-intercept. Moreover, the post-test results show that many of the students continued to use the x-intercept even after the discussion sessions (50% used the x-intercept at

---

[1] The transformation of students' descriptive language is discussed in detail in my doctoral dissertation (Moschkovich, 1992).

[2] The results of the written assessments and the analysis of the videotaped discussion sessions are discussed in detail in my doctoral dissertation (Moschkovich, 1992).

least once in the post-test). I will argue that coming to understand the meaning of equations of the form y=mx+b, the role of the parameters m and b, and the convention of not using the x-intercept for this form is a more complicated process than accepting a convention, erasing an error, or contradicting a misconception. First, through an analysis of the subject matter informed by the analysis of the data, I will discuss how the use of the x-intercept is a reasonable conception which reflects the mathematical complexity of this domain and exposes what is problematic about the connections between the graphical and algebraic representations. I will then show that it is a useful conception which helps students not only to produce an answer, but which is also correct in certain instances and briefly summarize how this conception was transformed. Lastly, I compare the misconceptions perspective with the perspective that some conceptions are transitional and explore the potential of this particular transitional conception as the focus of classroom discussions.

**Overview of the Study**

The study utilized videotaped data from peer discussion sessions after school. The eighteen students who volunteered to participate in the discussion sessions were from the two classrooms observed earlier in the school year (Moschkovich, 1990). Students completed written assessments immediately before and after the discussion sessions.The responses on the pre-test and post-test were used to document the extent of particular student conceptions and to assess any improvement after the discussion sessions. The videotapes of the peer discussions were analyzed to document the existence of specific conceptions about straight lines and their equations and to describe the transformation of one specific conception.

**Table 1: Data Sources**

**Pre-test**

Duration: Approximately one hour

Immediately following the chapter on quadratic functions and immediately preceding the discussion sessions.

Data: 18 students completed 28 questions

**Peer Discussion sessions**

Duration: There were no time constraints

Sessions lasted from two to four hours over a period of at least two days and at most four days.

Data: Videotapes of all sessions

**Post-test**

Duration: Approximately one hour

One to two days after the discussion sessions

Data: Same as the pre-test

Inherent to the design of the study is the triangulation of both data sources and methods of analysis (Guba and Lincoln, 1985). Working hypotheses developed during the classroom observations were verified through the analysis of the written and videotape data, and refined by longer term methods of analysis. The central arguments of this paper were thus developed through the "constant comparative method" (Glaser and Strauss, 1967) across settings, data sources, individual case studies, and analysis methods.

## Mathematical complexity of the subject matter

The task for the students in this study was to make sense of the connections between the algebraic and graphical representations of linear functions. Using the x-intercept for equations of the form $y=mx+b$ is a reasonable conception to include in this process. In effect, it reflects the mathematical complexity of this domain[3]. I have thus far used the term "x-intercept" to refer to either the point where a line crosses the x-axis or the number which is the x-coordinate of this point (and which can be used in an equation). For the sake of clarity, during the following discussion I will continue to use the terms "y-intercept", "x-intercept", and "slope" as general terms for the objects in either representation. However, I will distinguish between the graphical and the algebraic objects as follows:

| | |
|---|---|
| $\mathbf{b_E}$ : the algebraic y-intercept | $\mathbf{b_G}$ : the graphical y-intercept |
| $\mathbf{m_E}$: the algebraic slope | $\mathbf{m_G}$: the graphical slope |
| $\mathbf{a_E}$ : the algebraic x-intercept | $\mathbf{a_G}$ : the graphical x-intercept |

The **m** and **b** correspond to the parameters in the form **y=mx+b**; **a** corresponds to the parameter **a** in the equation **x/a + y/b=1**, where **a** is the the x-coordinate of the x-intercept, and **b** is the y-coordinate of the y-intercept. The subscript E stands for "Equation" and the subscript G stands for "Graph".

There are differences in how perceptually salient the slope, the y-intercept, and the x-intercept are as well as what steps are necessary for obtaining information from and about these objects. Thus, these objects have a different status within each representation and in relation to each other. In the graph of the equation y=mx+b, $\mathbf{b_G}$ is a special point, accessible directly by looking at the graph and locatable by its ordered pairs. On the other hand, $\mathbf{m_G}$ is only accessible graphically after estimating the slope or determining the rise and the run. While $\mathbf{b_G}$ is one point on the line, $\mathbf{m_G}$ is an object which is a characteristic of the line as a whole. When connecting the two representations, while $\mathbf{b_E}$ is the coordinate that pairs up with an x-coordinate of 0, $\mathbf{m_E}$ does not correspond to one coordinate. In contrast, in the equation $\mathbf{m_E}$ and $\mathbf{b_E}$ do have similar status. They are both letters, and although one object is related to the variable x by multiplication, and the other by addition or subtraction, this representation in no way gives one a clue to their different graphical status.

In terms of the two intercepts of a line, on the graph $\mathbf{b_G}$ and $\mathbf{a_G}$ have the same status: they are perceptually salient and directly accessible by their coordinates. However, in any equation not of the form x/a + y/b=1 (and even in that form the coordinates of the intercepts are not exactly easily accessible), the two intercepts do not have the same status in the algebraic form that they have on the graph. In the form y=mx+b, $\mathbf{b_E}$ is directly accessible in the equation, while it is necessary to set y equal to zero and solve for x to arrive at $\mathbf{a_E}$= -b/m.

[3] Although the following is an analysis of the subject matter from a competent perspective, it was inspired in large part by the observations of students grappling with this domain.

The conceptual asymmetry between the two representations is that not all the information accessible from the graph shows up in any one equation form. Moreover, not all the information that is available graphically is necessary for determining a line algebraically through any one form of the equation. For any of the available algebraic forms, the mapping between the algebraic and graphical objects is by no means simple. Thus, this conceptual asymmetry exists regardless of which form of the equation is used. For equations of the form y=mx+b, this conceptual asymmetry is evident in the x-intercept: while the x-intercept has status as a graphical variable, it does not as an algebraic variable in this form of the equation. Since $\mathbf{a_G}$ is as perceptually salient as $\mathbf{b_G}$ it is not surprising, then, that students would expect it to be as easily accessible in the equation as the y-intercept is.

While $\mathbf{a_G}$ and $\mathbf{b_G}$ are points, $\mathbf{m_G}$ is a more global object. For some students the concept of intercept may not include the fact that one of the coordinates is 0, and thus they may regard the intercepts as one-dimensional objects. In the case of the slope, it is necessary to see the line as an object which changes its inclination in a two-dimensional plane. If the choice is between using one two-dimensional object (slope) and one one-dimensional object (y-intercept), or two one-dimensional objects (the intercepts), the choice is clear from a simplicity standpoint. Even if students do not see the intercepts as one-dimensional but as points in a two-dimensional plane, the intercepts are still simpler objects than the slope, both perceptually and algebraically, since slope is neither a point nor an extensive quantity. It is not surprising, then, that students would choose to focus on the x-intercept over the slope. Simply on the basis of the algebraic and graphical status of these six objects, students' attempt to use the x-intercept in the form y=mx+b is a reasonable conception.

Furthermore, since the x-intercept changes as either the slope or the y-intercept change, it is also reasonable that students would focus on the x-intercept and have difficulties specifying the relationships among these objects. Changing either $\mathbf{m_E}$ or $\mathbf{b_E}$ has a graphical effect not only on $\mathbf{m_G}$ and $\mathbf{b_G}$ but also on $\mathbf{a_G}$. The use of the x-intercept for $\mathbf{m_E}$, then, is also a reflection of the dependence of the x-intercept on the slope. The x-intercept does appear in the form y=mx+b but not in a simple or directly accessible way. The very subtle simplifying assumption that we will not consider the x-intercept for this form of equations is neither explicit nor simple.

Developing a conceptual understanding of linear equations and their graphs means developing a perspective which involves much more than accepting the simple convention that the form y=mx+b involves only the parameters m and b, or that the x-intercept would only be accessible algebraically in other forms of this equation (such as x= y/m - b/m or x/a +y/b=1). Developing this perspective means unpacking how the parameters in the equation are manifested on the graph, how objects on the graph are manifested in the equation, and which objects in each representation are independent. Thus, the use of the x-intercept is not merely the result of choosing or emphasizing the form y=mx+b over other forms but is, instead, a reflection of the mathematical complexity of this domain.

**Illustrations of students' use of the x-intercept**

In this section I will present data from two case studies focusing on students' use if the x-intercept. These case studies show that the use of the x-intercept is a conception which was invoked even by the most capable students, and thus is not simply related to students' ability or achievement in mathematics. Moreover, the use of the x-intercept is not always wrong. First, it can be used to compare two lines; second, the opposite of the x-coordinate of the x-intercept can be used to generate $\mathbf{b_E}$ when m=1; lastly, it can be a useful way to explore the slope.

**Case Study 1: FR and HE**

These two students had the highest scores on the pre-test (29/31 and 23/31) and gave the best explanations for the pre-test question targeting the use of the x-intercept (m=1). Even though these two students also answered the discussion problem targeting the x-intercept (m=1) correctly, and did not use the x-intercept for two other problems with slope 1, they did invoke the x-intercept when working on a problem where the slope was not 1. During their discussion of the first problem with slope 2 they were asked to change the equation for one line on the graph (y=2x) to an equation which would produce the other line on the graph (y=2x+6, where $\mathbf{a_G}$ is (-3,0)).

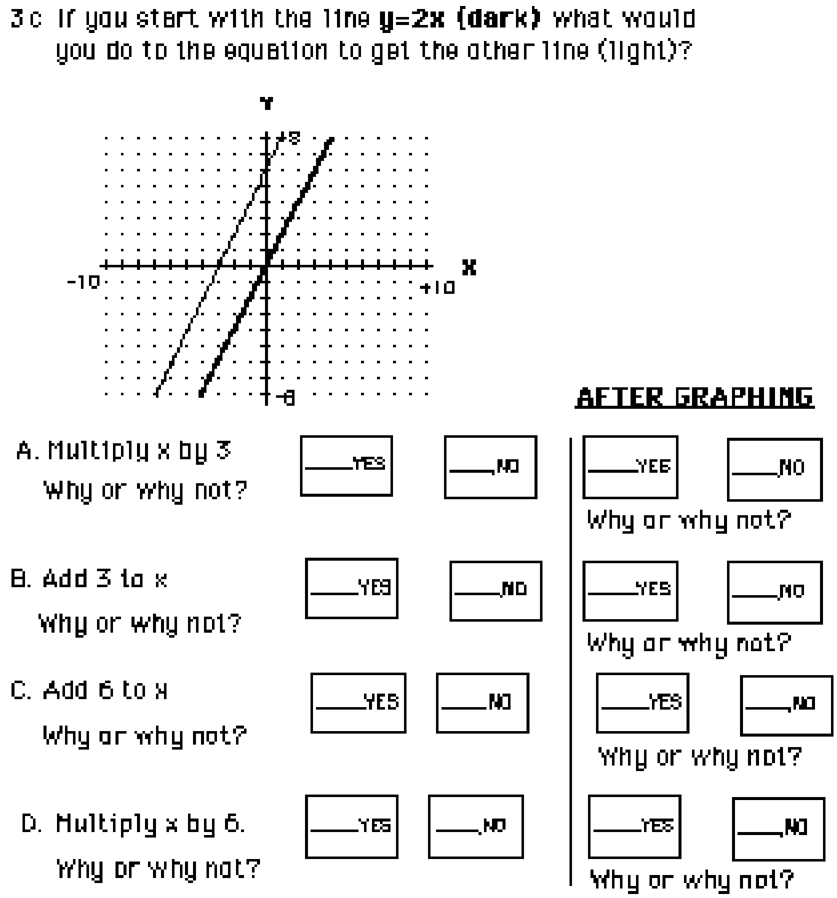
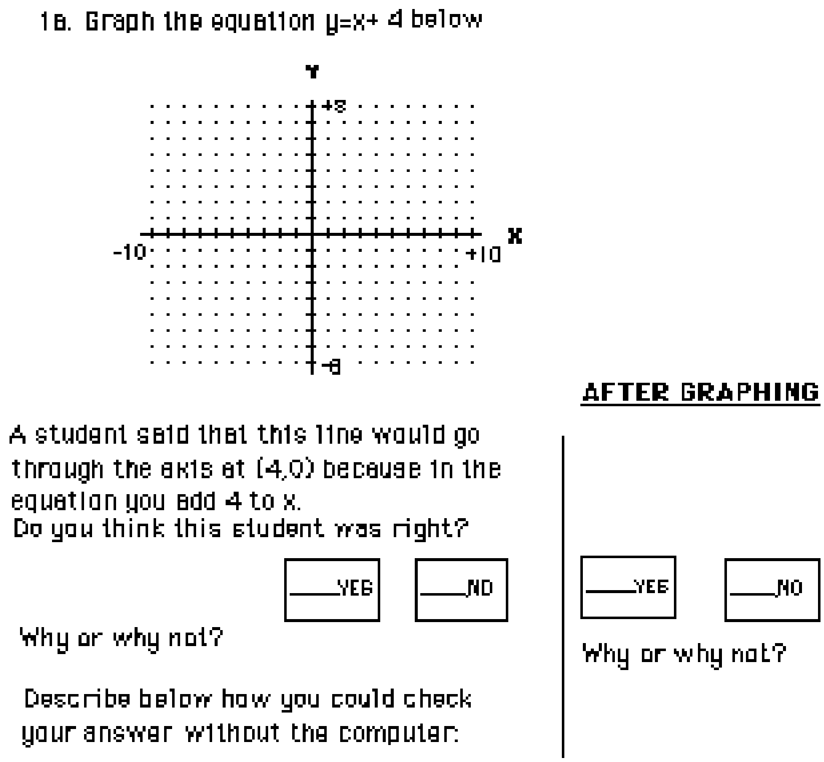



**Figure 1: Problem 3c**

Their first attempt was to use the opposite of the x-coordinate of $\mathbf{a_G}$ for $\mathbf{m_E}$. Their second attempt was to graph the equation y=2x+3, using $\mathbf{-a_G}$ for $\mathbf{b_E}$, which may have been a use of the x-intercept which had worked for the previous problems where the slope was 1. After several other attempts, they arrived at the equation y=2x+6. When they later worked on the discussion problems targeting the use of the x-intercept, but where the slope was 1, these two students had no difficulty answering that $\mathbf{b_E}$ did not correspond to $\mathbf{a_G}$ and explaining why this was not the case using the connection between the ordered pairs, the equation, and the line. In the case of Problem 1a, where the slope is 1, they recognized that the equation y=x+4 does not go through (4,0) and provided the following explanation:

**Figure 2: Problem 1a**




**Excerpt 1**

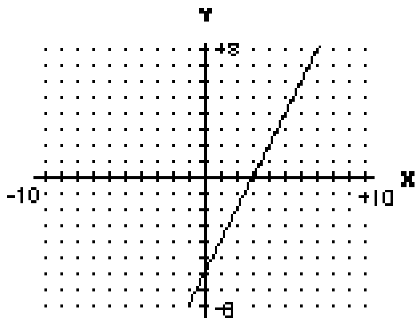
**FR and HE: Problem 1a**

1. HE: Graph the equation . . . (4,0) . . . it don't go through (4, 0) . . .
2. FR: Go through (4, 0) no . . . it doesn't go through.
3. HE: Cause it doesn't go through because . . .
4. FR: Go through x , right? . . . it doesn't go through x.
5. HE: It goes through x . . . but it says (4,0).
6. FR: This is x . . . it doesn't go through x
7. HE: No it doesn't go thru (*mumbles*). . . it says 4,0 . . . it doesn't go 4 and up 0.
   FR: Go up you . . .
8. HE: Because it doesn't move up . . . right 4 places and go up zero. (*Writes this*
9. *answer down*.)

It would seem as if by this point they understood that the x-intercept is not $\mathbf{b_E}$. However, this did not mean that they understood the meaning of the y-intercept in the equation. When trying to generate the equation for the line y=2x-6, in the next problem, they tried to use the y-coordinate of the y-intercept for $\mathbf{m_E}$ (line 4, and lines 6-8). This is most striking in that by this point they had worked on 21 problems (most of them using the y-intercept where m=1), and they had already discovered that to change the line y=2x to the line y=2x+6 one needed to add 6 in Problem 3c:

**Figure 3: Problem 1b**

1b. Write an equation for the line graphed below.




AFTER GRAPHING

| | |
|---|---|
| EQUATION:________________ Why? | EQUATION:____________ Why? |
| A student said that the equation for this line was y=2X+3 because the line goes through the x axis at x=3. Do you think this student was right? ___YES ___NO Why or why not? Describe below how you could check your answer without the computer. | ___YES ___NO Why or why not? |

**Excerpt 2**

**FR and HE: Problem 1b**

1. FR: This will be . . . 1,2,3 . . . this is hard!
2. HE: This is y equals . . . 1,2,3,4,5,6.
3. FR: What will the equation be?
4. HE: Negative six x . . .
5. FR: How will we find the equation? . . . what the equation is . . .
6. HE: Y equals . . . negative 6x . . . (*Silence*.)
7. FR: Negative six x?
8. HE: Negative six x . . . do you want to try it?
9. FR: Didn't we have one like that let's go back here. Find the one with two x . . . This is negative six x.
10. HE: Negative six x ( *mumbles*).
11. FR: Yeah this one . . . two x . . . 2x+6? (*They look back to previous problem, 3c, where the equation is y=2x+6*.)
12. HE: Plus six it would go up six.
13. FR: So it's . . .
14. HE: Going down six . . .
15. FR: Negative two x minus six to get it here . . . oh, two x minus six then
16. HE: Two . . . 2x+6 . . . .this one is the opposite of this so it's minus six, let's try it . . . (*Writes the answer down for Problem 1b*.)
17. FR: Let's write the equation down . . . 2x-6 . . . 1,2,3 (*counts 1,2,3 on x-axis*) 2,4,6 (*counts from -1 to -6 on the y-axis*) . . . .Yeah . . . why . . . . . Because . . . .I'm not sure . . . .

In summary, even these two students who seemed to understand the x-intercept on the pre-test used the x-intercept during problems where the slope was not 1. Thus, understanding and using the connection between an equation and a line (that the point (3,0) does not satisfy the equation y=x+3) does not preclude that a student may still use the x-intercept in another context. The use of the x-intercept is a conception which was invoked even by the most capable students, and thus is not simply related to students' ability or achievement in mathematics. This case study also shows that while students may seem to understand the x-intercept in one context (a problem with slope 1), they may use the x-intercept in other contexts (problems with slopes other than 1).

**Case Study 2: MT and JS**

While these two students scored very low on the pre-test (1/31 and 1/31), they showed substantial improvement on the post-test, both in terms of their overall scores (MT: 29/31, JS: 23/31) and , in the case of MT, in terms of his use of the x-intercept. While on the pre-test MT used the x-intercept in three instances for b, in one instance for m, and in three instances to describe line movement, there was no evidence of any use of the x-intercept in his pos-test responses.

During the first discussion problem MT and JS provided an explanation of why they expected $\mathbf{b_E}$ to correspond to the x-coordinate of $\mathbf{a_G}$. When predicting what the graph of the equation y=x+4 would look like (Problem 1a, Figure 2), MT expected to generate lines of the form y=x+b by starting from the line y=x and moving to the right along the x-axis:

**Excerpt 3**

**MT and JS: Problem 1a**

1. MT: "The student said the equation would go through the axis at (4,0) (*reading the question*)." Yes, why? Because in the equation you add 4 to x . . . now graph it? . . . x plus four (*Graphs the equation y=x+4.*) . . . what?
   JS: Oh, oh!
2. MT: I don't get this . . . .hmmm.
3. JS: I forgot that it goes on this side (*pointing to the II and III quadrants*).
4.
5. MT: This is single x . . . .this is x (*traces the line y=x with his hand*), right? Right through the middle is x . . . so this should be x one, x two, x three, x four (*successively pointing to lines through (1,0), (2,0), 3,0), (4,0)*).
6. JS: What about negative four? It might be over there (*pointing to (4,0)*), just like this (points to their graph on worksheet).
7. MT: This is the positive side (*pointing to the positive side of the x-axis*) . . . that's negative 4 . . . looks like that (*referring to the line on the screen, y=x+4*) would be y equals . . . x minus four . . . hmmm . . . Want to try to make it look like that (*points to the paper which is not visible*)? Let's see what we would get . . . all right we'll try . . .
8. JS: One x . . . one negative x plus one.
9. MT: Oh . . . OK! Yeah, see! By having it cross through here (*points to the paper, so it is either the point (0,-4) or the point (4,0)*) and that being negative . . . .huh . . . .Wait (shakes his head). Try x minus 4 . . .
10. JS: Hope it comes out right. (*They graph the equation y=x-4.*) Yep . . .

Starting with the line y=x, MT expected to generate lines of the form y=x+b by moving to the right along the x axis (line 5). His perspective on the equation y=mx+b is heavily influenced by the line and equation y=x, which is a very salient graphical object. This line is easy to recognize on the graph. When changing the equation from y=x to y=x +b it is possible to parse the equation y=x+b as:

**y=x +b** or **y=x (+b)**

In the second parsing y=x and (+b) are each algebraic objects. In the graphical representation y=x is a line, but b is not a line, it is a coordinate. When generating the line y=x+4 from the line y=x, one interpretation of the relationship between the two equations as "take each point on the line y=x and add 4 to each x-coordinate", an interpretation which corresponds to the the first algebraic parsing. Alternately, it is also possible to interpret the relationship between the two equations as "take the line y=x and add 4", which corresponds to the second algebraic parsing. MT's explanation in Excerpt 3 above reflects the second parsing.

MT further specified that he expected the sign of $\mathbf{b_E}$ to be the same as the sign the sign of $\mathbf{a_G}$. After JS suggested that the line for y=x-4 might cross the x axis at (4,0) (line 6), they graphed this equation. MT was then puzzled to see that a plus 4 in the equation would appear as an x-coordinate of minus 4 for $\mathbf{a_G}$:

**Excerpt 3 Continued**

**MT and JS: Problem 1a**

11. MT: Now this I don't get . . . .this I don't get at all . . .
12. JS: They're parallel . . . see the only thing that's negative is this (*pointing to the y-intercept of the line y=x-4*).
13. MT: I know! . . . .This is x (points to origin), if it's taken away 4 why would it go this way (*pointing to the point (4,0)*) and not that way (*pointing to the point (-4,0)*)?

They concluded that $\mathbf{b_E}$ corresponds to the opposite of the x-coordinate of $\mathbf{a_G}$ which is correct for lines of slope 1. They went on to use this conclusion in their first attempts during the next problem (Problem 1b, Figure 3, where the slope was 2:

**Excerpt 4**

**MT and JS: Problem 1b**

1. MT: What equation would this be? 3 . . . . x minus 3 . . . cause that one was x minus 4.
2. JS: X negative 4 . . . negative 3 . . . you gotta remember about this (*pointing to the paper which is not visible*.)
3. MT: What?
4. JS: You gotta turn it.
5. MT: The angle, huh?
6. JS: That looks like a 5 or 6.
7. MT: No, that would be more up so we want to say 5x minus 3? y . . . 5x . . . OK (*writing the explanation on the paper*). We used 5x for the angle and -3 for the line position. Ok, let's try it.
8. JS: (*They graph the equation y=5x-3*.) It's getting closer.
9. MT: Five is too big of an angle.
10. JS: This one's more upright . . . Wait a second wait, flip it over again.
11. MT: Three x (*they graph the equation y=3x-3*) . . . huh . . . .
12. JS: Where does it pass through five? In the first one it did . . . that looks like it passes through four.
13. MT: Yep . . . this (*pointing to the line on the worksheet.)* passes thru six.
14. JS: Aieee . . . Maybe y equals . . . yikes!

In the excerpt above they used the x-coordinate of $\mathbf{a_G}$ for $\mathbf{b_E}$ (lines 1,2, and 7). They went on to discover that in this case $\mathbf{b_E}$ did not correspond to the opposite of the x-coordinate of $\mathbf{a_G}$ but to the y-coordinate of $\mathbf{b_G}$:

**Excerpt 4 Continued**

**MT and JS: Problem 1b**

15. MT: Point five or so x . . . point five, let me try . . . maybe that will come along and go in there *(pointing to (0,-6) on the screen. They graph the equation y=0.5x +3*) . . . Whoa . . . .hmm
16. JS: You think it ignored the point 5?
17. MT: No, that's coming (*traces the last line graphed from the right edge of grid)*, that's x . . .
18. JS: That's three right there *(pointing to the common y-intercept for all the lines on the screen)*.
19. MT: Maybe . . . it's gotta be something higher than one *(referring to the slope number)*. We tried 4, didn't we and what's the lowest we tried . . . three?
20. JS: Oh, wait a minute, how about 3x minus 5?
21. MT: Three x minus five? (*They graph the equation y=3x-5.*) . . . Nope . . . x is too high.
22. JS: It went through the five.
23. MT: But we have to go down six.
24. JS: How are we gonna move it over here (*pointing to (-3,0))* that looks like 2,4,6,8,10 yeah . . . . How we gonna move it over?
25. MT: One point five x minus six . . . so y equals one point five x minus six . . . Hopefully . . . (*they graph the equation y=1.5x-6*). Yep.
26. JS: Gonna make six.
27. MT: Not quite . . . it did it . . .
28. JS: I think it ignored it . . .
29. JS: No, it didn't, it did, these things do it in steps . . .
30. MT: It went down through six.
31. JS: Steps of two (*referring to the units on the graph*) . . . this is one point five . . . it made it through six.
32. MT: Yeah, but this we gotta get to go up to three . . . So what do we try, two x?
33. JS: All right. (*They graph the equation y=2x-6.*) . . .
34. MT: Yeah, that did it. (*Writes the answer for after graphing, y=2x-6, on the paper.*)
35. JS: But how does it work?
36. MT: How does it work?
37. JS: Yeah . . . Cause I know this hit six (*points to the y-intercept for the line y=2x-6*) but how do you know what to . . .

Next MT and JS went on to explore the relationship between $\mathbf{a_G}$ and $\mathbf{m_E}$ by graphing several equations, y=-3x+6, y=6x-3, and y=3x+6, thus using either the x-coordinate of $\mathbf{a_G}$ or its opposite for $\mathbf{m_E}$ and for $\mathbf{b_E}$. They began by using $\mathbf{a_G}$ for $\mathbf{m_E}$ in the equation they produced (line 38):

**Excerpt 4 Continued**

**MT and JS: Problem 1b**

38. JS: Negative 3 x plus . . . plus 6 or what do you think?
39. MT: Yeah let's try that oh, we gotta do that first . . . y=-3x+6?
40. JS: Yeah.
41. Int: Did you answer why it's negative three x plus six?
42. JS: Cause I meet the coordinates.
43. MT: We got the negative three x cause the line was back three (*pointing to the x-axis on the worksheet)*.
44. Int: OK.
45. MT: And we got the plus six cause it ran through the six on the y-axis.
46. JS: OK, want to try it out?
47. MT: Yeah . . . (*They graph the equation y=-3+6*.)
48. Together: Oh . . . .Oh!!!!
49. JS: Let's take a look at this . . .
50. MT: Oh brother! Let's try . . . let's do this . . .
51. JS: Positive six plus . . . positive.
52. MT: Let's try the student he said this six x minus three (*they graph the equation y=6x-3* .) Nope . . .
53. JS: OK, ah . . . let's try positive x plus six . . .
54. MT: How about just x plus six?
55. JS: Positive three . . .
56. MT: Three x plus six?
57. JS: Yeah, cause maybe we put it on the wrong side.
58. MT: Yeah, that might do it. (*They graph the equation y=3x+6* .) . . . Well, almost did it.

It is clear from their explanations in lines 38-47 that they were in effect using the y-coordinate of $\mathbf{b_G}$ for $\mathbf{b_E}$ and the x-coordinate of $\mathbf{a_G}$ for $\mathbf{m_E}$. Using the x-coordinate of $\mathbf{a_G}$ for $\mathbf{m_E}$, while incorrect, reflects the fact that the x-intercept does change as $\mathbf{m_E}$ changes. The fact that changing $\mathbf{m_E}$ has an effect on $\mathbf{a_G}$ makes it difficult to separate the two objects or discover that $\mathbf{m_E}$ is not manifested on the line as the x-intercept. Thus, even for these two students who realized that the x-intercept does not correspond to $\mathbf{b_E}$ in the equation, the relationship between the x-intercept and the slope was difficult to unravel. In the last part of their discussion of Problem 1b, MT and JS began to specify the relationship between the slope and the x-intercept (line 71). Moreover, they were also treating the slope and the y-intercept as independent objects:

**Excerpt 4 Continued**

**MT and JS: Problem 1b**

59. JS: We have to put it here (*pointing to the point (-3,0)*) right?
60. MT: Yeah . . . so maybe 3.5x? . . . 3.5 x plus 6 . . . (*They graph the equation y=3.5x+6*.)
61. JS: It's even worse . . .
62. MT: Whoa . . . we gotta go higher instead of lower . . . like four or five x . . .
63. JS: Two . . . two . . . . how about two?
64. MT: No, two will bring it here (*uses a pen to show a line through the point (-1,0)*).
JS: No . . . we raised it (*referring to the slope number*) . . .
65. MT: That's right we did raise it (*referring to the slope number)* . . . So we gotta go
66. lower.
67. JS: Want to clear this off (*referring to the screen*)?
68. MT: No it doesn't' matter, . . . we got enough to do . . . plus six.
69. JS: Plus six. (*They graph the equation y=2x+6*.) . . . made it . . .
70. MT: Yep . . .
71. JS: Why do you have to do two just to get three?

Starting from the line for y=3x+6, they tried to change the x-intercept by using different numbers for $\mathbf{m_E}$. Since the x-intercept of the line y=3.5x+6 is to the right of the the x-intercept of their previous line, y=3x+6 (lines 59-61), and they had wanted to move the x-intercept to the left, they decided to try a smaller number (line 66). Finally they used the number 2 for $\mathbf{m_E}$ (lines 63-69) and produced the correct equation.

In summary, MT and JS's exploration of the slope as a reflection of the x-intercept shows that even after discovering that the x-intercept does not correspond to $\mathbf{b_E}$ students continue to see the x-intercept as a relevant object because it is a marker for changes in the slope. Furthermore, during their discussion of Problems 1a and 1b these two students moved back and forth between using the x-intercept for $\mathbf{b_E}$ and for $\mathbf{m_E}$, depending on the problem context. For Problem 1a, this was a useful application of the x-intercept, while for Problem 1b, where the slope is 2, this use of the x-intercept is not applicable.

**Summary**

The two case studies discussed above show that students' use of the x-intercept is context dependent, rather than simply right or wrong. In some contexts it is useful, in some contexts it is applicable, while in other contexts it is not. First, using both intercepts to compare lines is an easy and direct way to check that the lines are the same. Many students used $\mathbf{a_G}$ and $\mathbf{b_G}$ to check that a line they had produced on the screen was the same as a line that was graphed on the worksheet. FR and HE (Case Study 1), for example, used either the x-intercept or both intercepts to check their lines 7 times for 21 problems.

Moreover, since the x-intercept reflects changes in the slope, the x-intercept can be used to explore the meaning of the slope (Case Study 2). Thus, there are at least two ways in which the use of the x-intercept can be refined. First, the contexts in which the x-intercept is applicable can be specified. Second, the use of the x-intercept can be refined from being considered as reflection of the y-intercept to being considered as a reflection of the slope.

Although I will not describe the refinement of this transitional conception in detail here, the analysis of the videotaped discussions for six pairs of students shows that the use of the x intercept was refined in the following ways:

- The use of the x-intercept for b when m=1 was refined from using the x-coordinate of the x-intercept ($\mathbf{a_G}$) as the b in the equation, to using the *opposite* of the x-coordinate of the x-intercept ($-\mathbf{a_G}$) as the b in the equation.
- The contexts in which the use of the x-intercept is applicable was specified. For example using the opposite of the x-coordinate of the x-intercept ($-\mathbf{a_G}$) for the b in the equation is applicable, but only when m= 1; another context in which the use of the x-intercept is applicable is when comparing two lines.
- The x-intercept was explored as a reflection of the slope.

**The use of the x-intercept as a transitional conception**

Students' use of the x-intercept presented here is a robust conception which reflects the complexity of the subject matter. The analyses presented above show that it is coherent, connected to other knowledge elements in the students framework, the result of a constructive process and that it originates in sense-making. The examples presented above show that the x-intercept is a transitional because it is a conception between two states developed as students are introduced to the domain and explore it further. This conception can serve as an entry point into the connection between algebra and graphs, the status of objects in each representation, and the dependence or independence of the objects in each representation.

The use of the x-intercept is a transitional conception which is: 1) between two states and progressing; 2) applicable, even if in limited contexts; and 3) can be transformed through refinement. The use of the x-intercept reflects the recent understanding that there is some connection between graphs and their equations, that if you change one representation there should be some change in the other. Future states are those in which this connection between the two representations is specified and refined in terms of which and how many parameters are necessary and sufficient to determine a line in two-dimensional space, how these parameters are related, and in which contexts the use of the x-intercept is applicable or not.

Using the x-intercept for the form y=mx+b is not always wrong. For example, it is useful for comparing two lines. Others times it is applicable, though in limited problem contexts. For problems where the slope is 1, the opposite of $\mathbf{b_E}$ will generate the correct x-coordinate for $\mathbf{a_G}$ when going from an equation to a line, or the opposite of the x-coordinate of $\mathbf{a_G}$ will generate the correct $\mathbf{b_E}$, when going from a line to an equation. Other times unpacking how the x-intercept is dependent on either the slope or the y-intercept makes exploring the two representations quite complicated. Thus, like other conceptions, the use of the x-intercept is context dependent.

While a transitional conception is robust, it is not rigidly held or solidly established. Students are committed to the use of the x-intercept because it is both reasonable and useful, and thus it keeps appearing. Students may oscillate among several uses of the x-intercept in the short term without being deeply committed to any one use. They may oscillate between two contradictory uses of the x-intercept within the same problem context (as was the case for Case Study 2, MT and JS) or invoke different uses of the x-intercept in different problem contexts (as was the case for Case Study 1,FR and HE). They may abandon a transitional conception in one context and return to it in another context. For example, students may not invoke the x-intercept for lines with slope 1 but use it for lines with slope 2 (case Study 1).

A transitional conception can evolve by refinement. One aspect of what is refined are the contexts in which the conception is applicable. Students' use of the x-intercept was transformed and evolved by redefining the relevant objects, the relationships between them, and the descriptions of these objects and relationships. In Case Study 2, the use of the x-intercept for m was refined to seeing the x-intercept as a reflection of changes in m. Thus, if the x-intercept is explored as a marker for changes in slope, it can serve as a bridge to the concept of slope.

**Misconceptions and transitional conceptions**

Understanding transitional conceptions has important theoretical implications. Within a social constructivist framework, misconception is no longer an adequate concept for referring to some of the conceptions that students generate. In particular, this concept was not adequate for describing or understanding these students' use of the x-intercept. Thus, the analysis of students' use of the x-intercept as a transitional conception highlights the shortcomings of the misconceptions perspective.

An analysis of the use of the x-intercept as a misconception view would see it as a mistake, go on to document it as a pattern of systematic errors, perhaps suggesting its origin in "prior instruction that students generalize" (Smith, diSessa, and Roschelle; in press). One of the assumptions of such an analysis would be that using the x-intercept for the form y=mx+b is wrong, regardless of context. As shown by the analysis presented in this paper, this is not the case. Another assumption would be that the use of the x-intercept is a mismatch with some principle of competent knowledge, such as "the x-intercept does not appear in equations of the form y=mx+b." The x-intercept does appear in this (or any other) equation form, just not in a simple or directly accessible way.

The subtle simplifying assumption that we will not consider the x-intercept for this form of equations is neither explicit nor simple. It is a complex assumption which reflects the accepted ways to work with equations in two variables. Thus, the seemingly simple questions "What happens to the line when you change b in the equation?," "What does the number added to x do to the line?," "What happens to the line when you change m in the equation?," or "What does the number multiplying the x do to the line?" are not simple at all unless one makes certain assumptions (or knows the answers). These assumptions include:

- It is not necessary to describe how all the objects that have graphical status will be manifested in the algebraic representation.
- Objects that have similar graphical status will not necessarily have the same algebraic status.
- Objects that have different graphical status may apparently have similar algebraic status (y-intercept and slope).
- Two parameters are necessary and sufficient for determining a line in 2-dimensional space or an equation in two variables.
- These two parameters are independent.

One last, and perhaps implicit, assumption that an analysis of the use of the x-intercept as a misconception would make is that this misconception will be replaced by the correct conception. Erasing the use of the x-intercept without paying attention to how this conception makes sense to students does not address how students make sense of the mathematics on their own terms. Since this conception is reasonable, useful, connected to other pieces, and not always wrong, it will continue to appear.

**Conclusions**

The data collected in this study show that there was a strong dependence on the x-intercept, despite the fact that the form of the equation students used most frequently, y=mx+b, does not employ or address the x-intercept. There were several uses of the x-intercept documented[4]:

- as the b in an equation;
- as the m in an equation;
- to compare lines;
- to describe line movement.

These uses of the x-intercept were dependent on problem context[5]. In particular, the value of the slope was relevant to whether the use of the x-intercept to determine b in an equation was applicable or not. Specifically, the opposite of the value of the x-coordinate of the x-intercept did indeed produce the correct value of b when m=1, but not when m≠1. A second problem context where the use of the x-intercept was applicable was when comparing or checking lines, and a third problem context was when the x-intercept was explored as a reflection of changes in the slope (rather than used as m itself).

The use of the x-intercept was analyzed as a transitional conception and shown to be reasonable, applicable in some contexts, and transformed by refinement. I also argued that the use of the x-intercept was not merely the result of choosing or emphasizing the form y=mx+b over other forms or a superficial error. Instead, it is a reflection of the mathematical complexity of this domain and the transitional nature of this conception. Changing the b in an equation does, in effect, move lines along the x-axis (as well as along the y-axis), so that descriptions which include horizontal translation are not necessarily wrong. What is important about these descriptions is that experts usually choose to focus on movement along the y-axis as result of changing b. This choice reflects the fact that lines move exactly b units up or down the y-axis when, for example, an equation is changed from y=x to y=x+b. While relating the

---

[4]Although the thesis documents several different ways in which students used the x-intercept, I consider the use of the x-intercept as a unitary conception, but which is manifested differently depending on the problem context. Thus, my analysis of the x-intercept is an analysis of a conception <u>in use</u>.

[5]I use the phrase "problem context" to differentiate between the larger social context and the particular problem a student is working on. The problem context includes the details of an equation, such as the value of the slope, and also how students define, interpret, and generate problems (for example, whether they are comparing two lines or exploring the slope).

parameter b in an equation to the movement of a line along the x-axis is possible, it is a more complicated correlation. Lines move either -b units along the x-axis, in the case of m=1, or (-b)/m units, in the case of m≠1. Focusing on movement along the y-axis reflects the simplest possible correlation between the two representations and is thus not an arbitrary choice.

The analysis of the use of the x-intercept as a transitional conception shifts the theoretical emphasis from misconceptions. The concept of transitional conception addresses, first, how a student conception makes "sense," but it also specifies whose "sense" we are talking about. The analysis of transitional conceptions also includes both positive and negative aspects of student conceptions. Analyzing how a conception is transitional includes a consideration of how a conception is right or wrong, better or worse, and more or less useful. However, these distinctions are not absolute but dependent on the problem context. For example, while using the opposite of the x-intercept to generate b is applicable for equations where m=1, it is not when m≠1. While seeing the x-intercept as a reflection of changes in the slope is useful as bridge to the concept of slope, simply using the x-intercept for m is neither applicable nor useful.

Inherent to an analysis of conceptions as transitional is a view of learning which assumes it is a continuous, as opposed to a disconnected, process. If we take constructivism seriously, the right conceptions must arise from already existing conceptions. If most incipient conceptions are simply wrong, regardless of the context in which they are used or the learner's experience with the use of mathematical tools, then learning constructively is an impossibility (Smith, diSessa and Roschelle, in press). Seeing learning as simply the replacement of the wrong conceptions by the right ones is thus problematic. In order to describe learning as a continuous process it is crucial to address how conceptions evolve.

Whether a particular conception is transitional or not for a particular student is not as important as taking the theoretical perspective that some conceptions have the potential to be transitional. The important question then becomes examining how a conception is transitional, instead of determining whether all conceptions are transitional or not. The evolution of conceptions, as shown in the case of the x-intercept, is likely to occur often. If transitional conceptions occur with some frequency, then it is important to focus on how they evolve, rather on how to eliminate them, and on supporting this evolution. Moreover, it is not possible to determine whether and how a particular conception satisfies the criteria for being transitional without carefully considering its origins, its uses in different contexts, and how it makes sense to students.

Changing our perspective of the nature and transformation of student conceptions has important implications for instruction. Instructional strategies and recommendations are based on different views of how student conceptions change. While analyses of misconceptions rarely address this question directly or explicitly (Smith, diSessa, and Roschelle; in press), they usually suggest instructional strategies which make implicit assumptions about the transformation of student conceptions. Thus, each view of student conceptions implies different pedagogical strategies. If student conceptions as seen as errors, then instruction is designed to remove these errors. If they are seen as misconceptions, then instruction is designed to contradict these misconceptions. If some student conceptions are seen as transitional, then instruction can address how these conceptions are reasonable and useful and exploit their potential for refinement.

Transitional conceptions, such as the use of the x-intercept, are precisely the kinds of ideas that appear during exploration and open-ended problems. If the instructional goal is to support students making sense of the mathematics (Schoenfeld, 1992), "correcting" students' use of the x-intercept is less useful than attending to how this conception makes sense. Once we begin to interpret some student conceptions as transitional and explore the underlying mathematical complexity they reflect, it is then possible to explore their potential as bridges to more competent conceptions.

**Exploring linear functions in the classroom**

Based on the results of the results of this study, classroom activities for exploring and discussing graphs and equations should address the conceptions that students generate such as the use of the x-intercept. The following suggestions for classroom activities illustrate how these criteria can be addressed by considering the sequence of examples and discussing student conceptions directly. First, a caveat. A classroom activity is not merely a problem as it is written on paper but how the problem is used in the classroom. Any of the criteria listed above can be used to design superficial exercises with a focus on the right answer. In order to encourage students to focus on the process rather than the answers, the activities should be introduced as problems for <u>exploring</u> and <u>discussing</u> lines and equations. The focus of the classroom discussion of these activities should be on conjectures and explanations, not "yes" or "no" answers. As part of these activities the teacher should initially model how to conduct a systematic exploration, construct explanations, and evaluate alternative conjectures. As exploration proceeds, the teacher can encourage the students to take on more of the responsibility for evaluating conjectures and explanations amongst themselves.

There are several ways to address students' use of the x-intercept. The selection and sequence of initial examples is an important consideration. Initial examples used for exploring y-intercept should include slopes other than 1. Moreover, if the examples use a slope between 0 and 1, the lines tend to look more as if they are moving up and down, rather than right to left

(See Figures 4 and 5). However, I am not suggesting that the choice and sequence of initial examples will eliminate students' use of the x-intercept. In effect, the nature of transitional conceptions is such that some transitional conception, whether it is the x-intercept or some other one, will be generated regardless of the choice or sequence of examples. Moreover, since the intercepts and the slope have different graphical and algebraic status, students can in general be expected to focus on the intercepts over the slope, even if initial examples have slopes other than 1.

Classroom activities can directly address and build on the use of the x-intercept. The point of addressing this conception directly is not, however, to erase it or contradict it. Rather, the goals should be to acknowledge it as reasonable, consider the problem contexts in which it is applicable, and use it as bridge to the concept of slope. The comparison of different problem contexts ($m=1$ and $m\neq1$) is therefore more important than the selection and sequence of these problems.

Since the x-intercept is perceptually salient in the graph of a line in a way that it is not in the equation, students may generate different predictions when starting from a line than from an equation. It is thus important to include activities which ask the students to translate a graphed line by changing a given equation. The fact that some lines look like they are moving right to left (or viceversa) while others look like they are moving up and down can also be an explicit focus of a class discussion. For example, given the two figures below, the teacher can pose the question:

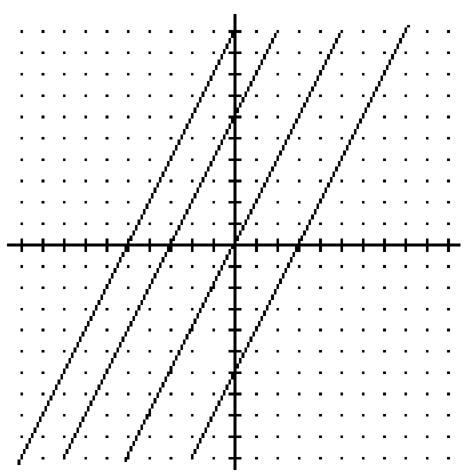


**Figure 4**

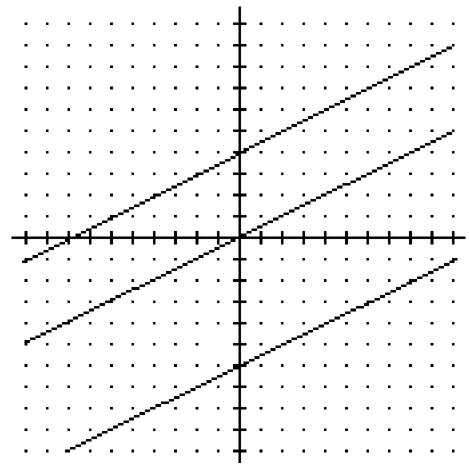


**Figure 5**

Some students said the lines in Figure 4 look like they are moving left but the lines in Figure 5 look like they are moving up and down.

Which do you think is a better description?

Why?

The point of activities such as the one above is to explicitly discuss the fact that describing lines as moving up and down as a result of changing the b in the equation is not an arbitrary choice, but related to the algebraic representation y=mx+b. The b in the equation is not only the y-intercept for each line. For two lines of the same slope, for example $y=mx+b_1$ and $y=mx+b_2$, the difference $|b_1-b_2|$ is also the vertical distance between the two lines (Figure 6). This is true at any point on the lines, not only at the y-intercept. Thus, the b in the equation y=mx+b has a direct graphical correspondence in terms of the vertical distance between two lines. The horizontal distance between lines, while it can be computed from the form y=mx+b to be $|b/m|$, is not as accessible in this form of the equation. Seeing lines as moving right and left, is not wrong, just less convenient.

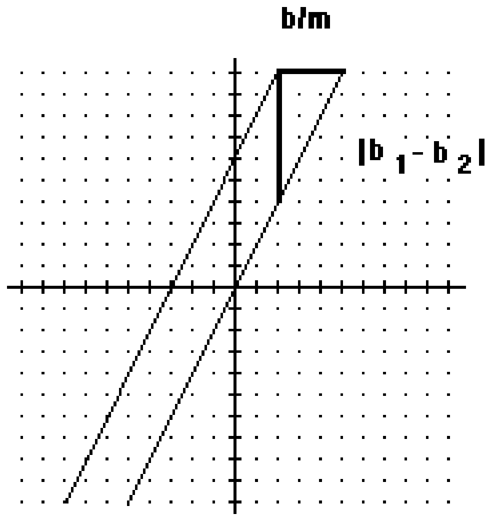



**Figure 6**

The x-intercept, like other transitional conceptions, can be used to involve students in mathematically productive discussions. Students can consider which of the conjectures they generate are applicable in which contexts and decide which conjectures are generalizable to several problem contexts. Classroom activities can thus address common transitional conceptions directly by making them the focus of discussions where students construct explanations and arguments.

**References**

Moschkovich, J. (1990) Students' interpretations of linear equations and their graphs. Paper presented at the Fourteenth Annual Meeting of the International Group for the Psychology of Mathematics Education, Mexico.

Moschkovich, J. (1992). Making sense of linear equations and their graphs: An analysis of student conceptions and language use. Unpublished doctoral dissertation.

Moschkovich, J. (1993). Making Sense of Graphs and Equations During Peer Discussions: The Negotiation and Transformation of Students' Descriptive Language. Paper presented at the 1993 annual meeting of the American Educational Research Association; Atlanta, April, 1993.

Moschkovich, J., Schoenfeld, A., and Arcavi, A. (1993). Aspects of Understanding: On Multiple Perspectives and Representations of Linear Relations, and Connections Among Them. In

T.A. Romberg, E. Fennema, and T.P. Carpenter (Eds.), Integrating Research on the Graphical Representation of Function. Hillsdale, NJ: Erlbaum.

Schoenfeld, A.H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), Handbook for Research on Mathematics Teaching and Learning. New York: Macmillan.

Smith, J., diSessa, A., and Roschelle, J. (in press). Misconceptions reconceived. To appear in the Journal of the Learning Sciences.

Schoenfeld, A.H., Smith, J.P., and Arcavi, A.A. (in press). Learning. To appear in R. Glaser (Ed.), *Advances in instructional psychology*, vol. 4. Hillsdale, NJ: Erlbaum.