Third Misconceptions Seminar Proceedings (1993)

Paper Title: SEARCH FOR MATHEMATICAL MODEL OF SCIENCE EDUCATION
Author: Ferbar, Janez

Abstract:

Keywords: theories,concept formation,teacher education,learning theory,linguistic theory,cognitive structures,cognitive processes,teaching for conceptual change,
General School Subject: physics
Specific School Subject:
Students: student teachers

Macintosh File Name: Ferbar - Science Education
Release Date: 12-15-1993 B, 11-5-1994 I

Publisher: Misconceptions Trust
Publisher Location: Ithaca, NY
Volume Name: The Proceedings of the Third International Seminar on Misconceptions and Educational Strategies in Science and Mathematics
Publication Year: 1993
Conference Date: August 1-4, 1993
Contact Information (correct as of 12-23-2010):
Web: www.mlrg.org
Email: info@mlrg.org

A Correct Reference Format: Author, Paper Title in The Proceedings of the Third International Seminar on Misconceptions and Educational Strategies in Science and Mathematics, Misconceptions Trust: Ithaca, NY (1993).

----- -----

SEARCH FOR MATHEMATICAL MODEL OF SCIENCE EDUCATION

Janez Ferbar, Pedagoska fakulteta v Ljubljani, Slovenija

Three basic activities in science are **classification, ordering, counting and measuring** of a set of objects. Classification is based on the equivalence relation and introduces qualitative variables. Ordering is based on the various ordering relations which are typical for semiquantitative variables. Counting and measuring are based on the operation of addition and introduce quantitative variables.

Sets with relational structure (=,>) could be mapped into language and into space when we distribute its members into space regions and give them names. Sets with operational structure (+) are mapped into positive integers by counting and into real numbers by measurement procedure.

Other types of mapping of one structured set into other are connected with the various **coordinating activities**.

This mathematical model of the basic science processes could serve as a basis for structuring the research in science teaching.

TERMINOLOGICAL CONSIDERATIONS

When reading about research in science education one is faced with the richness of terminology being used. In the first developmental stages of a new field this abundance is functional because it helps to make new associations. Later on it becomes disfunctional because it does not provide for more refined distinctions. Terminology with high redundancy does not easily converge to some mathematical model which is usually desirable in every scientific endeavour.

Classification is often looked upon as the basic act of cognition (Nelson K, in Johnson - Laird P C et al, Thinking). This activity itself has many names like: categorisation, grouping, sorting out.

Its result it as least one subset of elements, which are in some respect equivalent to each other. So we can use mathematical terminology and call it equivalence class. In literature equivalence class could have different names like: category, group, concept, collection, heap, spatial group. The classification can only be started in a set of elements which differ in some respect. They have a least two attributes.

One is common to each of them, it is constant in the set and differentiate the elements of the set from non-elements.

The other must take at least two different values in this set. So it is variable in the set and makes possible to differentiate among members of the set and to classify them. In the case that members of the set to be classified are physical objects this is a discontinuous (discrete) spatial variable. It is very common to limit classification experience of children to physical objects alone.

The classification procedure yields a new property ascribed to those members which are also members of the newly formed equivalence class.

If there exist an equivalence relation among the members of the rest of the set, this equivalence class defines another property inside the original set. Because both equivalence classes were obtained by the some procedure they define two **values** of the some binary **variable** in the original set.

Variable is called differently in the current literature: principle of categorisation, basis of categorisation, concept, attribute, criteria attribute rule, dimension, criterium of categorisation, basis of classification, quality, state, relationship etc.
All these expressions could be found even in a single article if its happens to be an overview of the field and if the reviewer does not wish to impose his or her own terminological preferences into the reviewed articles.

OTHER CLASSIFICATION REFERENTS

Of course it is also possible to classify **events**. They are characterised by their specific **time variables** unlike objects whose basic attributes is **time constant**. The events can be classified by variables but also by the constants which are conserved when some other attribute of the observed system is being changed. By classifying events we get properties of events.

In language events are usually described by verbs. In the simplest case this can be done by perfective verbal forms. In mathematical terms this can be replaced by an ordered pair of initial and final values of the definitional variable. For instance: perfective verb to stretch can be replaced by ordered pair (short, long), which are initial and final values of the variable length. The final value could be obtained from the initial one with the time integral of the event. So perfective verbs can be understood as a linguistic counterpart to integral description in mathematics. Similarly it is possible to argue that imperfective verbal forms are predecessors of differential description in mathematics.

Classifying events means classifying ordered sets. If several events have the same initial state then classification of events can reduce into the classification of the final states. This reduction of complexity is a useful didactic device.

It is also possible to classify properties and relations what yields properties of relations and properties of properties. In mathematical terms this means classifying sets of sets, sets of ordered pairs (representing relations) and sets of sets of sets. This possibilities introduce higher order classifications. The possibility that elements of a set could be sets leads to **ordered structure**. Ordering relation in this case is the relation of inclusion.

OPENING QUESTIONS

Until now we have not dwelled on the classification procedure. The debate about primacy of classificatory variables has a long history(7).

Which type of variables are used by children first: functional or perceptual? Functional (of course) seems to be common agreement nowadays. But there are other questions which are still open. The most common referents of classification procedure are objects and their sets. Much less has been said about classifying properties, states, relations and events. Even classification of substances according to their intensive attributes have been very much neglected in the educational literature.

There are also many ways for establishing the equivalence relation on which the classification procedure is based. It can be subjective interaction with the set of referents or an objective interaction in which the classifier is reduced to a (noninteracting) observer. An example for subjective classification is tasting various soups. Example of an objective classification is the partition of a set of things into floaters and nonfloaters by throwing them into water.

Some classificatory procedures are unique. They are being used for operational definition of properties. More often it happens that different classificatory procedures lead to the same partition. The properties defined by such procedures are **interdependent**. This interdependency can vary from identity to functional or probabilistic interdependence. The classificatory interactions are then also connected with each other. Quite often one of them can serve as an **explanation** for the others.

Perceptual classifications have a very prominent role. Many of them are nondestructive and easily done. So it is very important if it is possible to relate observable properties with the properties relevant for different objective interactions. Just by looking to the crystal glass and perhaps by listening to the sound of it if touched by spoon one can predict what will be the results of interaction of this glass with the concrete floor underneath.

Until now I have mostly discussed the cases in which the classificatory attributes were **time constants**. If the referents of classification were objects then the classificatory attribute were spatial variable. The events however could be classified in accordance to the characteristic **time variable**. If the events are taking place in different regions of space we have to deal with space and time variables at the same time. Various reduction procedures are used to simplify the task.

The simplest way to classify the events is classification by their outcomes if they start from the same initial conditions. This is the first step to the introduction of time variable. Probably time variables are often introduced on the basis of previous experience with the same attribute being introduced as a space variable. Putting different weights on the same spring introduces length as a time variable. The same space variable has been introduced much earlier when classifying objects by their length.

WHY CLASSIFY

Sets differ from collection of things by the fact that the elements of a set are related by the **same** equivalence relation. No such unifying relation exists in a collection of things in a drawer "for everything". Because the elements of a set are all equivalent to each other in one respect they can be mutually substituted by each other in those events in which the definitional property of the set influence the outcomes or the development of event.

Classification is a powerful tool for simplification of the world. Binary classification for instance reduce a set of n elements into only two different classes. To gain an insight about possible outcomes of an event in which the classificatory variable is important it is necessary to experiment only with one representative of each equivalence class. The events with other class members will give the same results.

The confusion has arisen from the Piaget's findings about relative deficiency of classification procedures among preschool children and Brunner's conclusion that school children progress from perceptual to functional classification.

These findings seem to be contradictory to the results of later research that infants engage in classificatory activities certainly in their second year well before they can give names to the equivalence classes which they form on the functional characteristics of the objects (i.e.what object do or what can be done with them). Perceptual attributes like colour do not seem to be possible classificatory variable for small children (up to 24 months). The best results have been observed with the sets of object which had redundant functional attributes, each of them leading to the same equivalence classes.

It has been also observed that children "sort out" from the set a single class and that they are not able to think about the attributes of the "rest". This is in accordance with much older findings. M. Montesori's teaching material is already designed so that **difficulties** are isolated. That means that members of a set differ only in one attribute.(It could be observed here that this careful insolation of only one variable does not seen to be functional for **naming** the objects. So it might be contraproductive in the language development.)

Others (Palmer and Rees, in Johnson - Laird P N, Watson P C (Eds) Thinking) have observed that children from 2 to 3 years better observe **time variables** (events) than **space variables** (local differences). This is functional for the survival because the ongoing event needs child's reaction **now**. Local differences do often cause the events but they are not sufficient condition for them and so they are not threatening by themselves.

CLASSIFICATION IS AN OPERATION ON A SET

The same authors (Palmer and Rees) have also found that there in no noticeable difference in the difficulties among the perceptions of the two major types of attributes: properties of the bodies and relations among them. Theoretical basis for those results includes "the model of a child who focuses on defining the relevant attributes of a single object and then generalises his concept definition to other similar objects (Nelson in Johnson - Laird P N, Watson P C (Eds) Thinking).

It is not clear how a child can discover any attribute without a **set** of bodies. Moreover it is logically impossible to discover which attribute(s) is/are **relevant** not mentioning the **event** for which it is relevant.

One can get more clear view if one starts from the premise that **all attributes** of various elements (objects, events, properties, relations) could be obtained by some sort of **classificatory procedure**. Logical necessity for classification is a **set of elements**. So one can do nothing with a single instance.

Further we have to recognise that classificatory procedure is a set of **events** in which two elements from two different sets take part — binary interactions. To enable classification it is necessary that the classificatory events differ in some respect or another. Often they start with equivalent initial states and end up in different final states. This is the only possibility for the classificatory procedure. It is simple however and let us stick to it.

In principle both elements in the interaction could be replaced by some other element from the same set. Let me call the two sets "subject set" $\{S\}$ and "object set" $\{O\}$ in the grammatical sense of the words. All possible binary interaction among the elements of the two sets could then be represented by the Cartesian product of the two sets with elements $\{(S_i, O_j)\} \equiv$ "S_i acts upon (interacts with) O_j."

	O_1	O_2	O_3	…	O_n
S_1	S_1, O_1	S_1, O_2	S_1, O_3	…	…
S_2	S_2, O_1	S_2, S_2	S_2, O_3	…	…
S_3	S_3, O_1	S_3, O_2	S_3, O_3	…	…
…	…	…	…	…	…
S_m	…	…	…	…	S_m, O_n

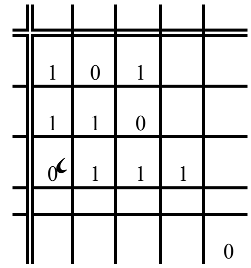
The interaction has in principle mn possible outcomes if m and n are numbers of subjects and objects respectively. If this is so no classes of more than 1 element could be formed. On the other hand all results could be equal and in this case neither $\{S_i\}$ nor $\{O_j\}$ could be partitioned into subsets by this procedure. Let us suppose that there are only two possible results R_1 and R_2 of the interactions. By performing all the interactions the Cartesian product is mapped onto the set of possible results $\{R_1, R_2\}$. The mapping fits the criteria for a function F if no interaction leads to both results:

$$R = F\ (S, O)$$

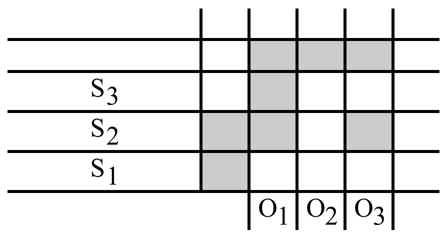
Let us assume that $R_1 = 0$ and $R_2 = 1$.

Experimentally defined function can be graphically represented in various ways:

	O_1	O_2	O_3	O_4	
S_1	S_1, O_1	S_1, O_2	S_1, O_3	…	…
S_2	S_2, O_1	S_2, S_2	S_2, O_3	…	…
S_3	S_3, O_1	S_3, O_2	S_3, O_3	S_3, O_4	…
S_4	…	…	…	…	…
	…	…	…	…	S_i, O_j



or more conventionally:



Last graph is threedimensional. The third dimension R has been represented by shading. The shaded ares is a graph of a relation from S to $O^{(3)}$. It might be observed that this **relation is time constant** if the classificatory procedure is **repeatable**.

Classification of objects could be done by any subject. The results classification done be S1 are shown in the lowest **raw**. The differences in the result are now ascribed to the objects; we ascribe them different values of a variable **property**. The experiments with the same object Oj can be used for classification of subjects.

These classificatory procedures could be used for two purposes at least: as **operational definitions** of properties of Ss or Os. Again those properties are time constants if the classificatory experiments are repeatable. Classification is possible only along the variable which is **relevant** for the classificatory experiment. Relevant variable is one that will produce different outcomes of experiment if its value will change.

It was said already that binary interaction can be used as a classificatory procedure einther for subjects or for objects. In the simplest case the interaction starts always with the same initial states. So the end states alone can serve the classificatory procedure.

If S is constant in such interaction different end states differentiate Os into equivalence classes. Usually the objects are looked upon as being passive. So their properties are often described with passive participles (broken, stretched) and adjectives describing passive properties like inertia, resistance, compressibility, charge.

If O is constant then Ss can be classified. Their attributes are usually defined as active. They are often described by active participles like driving, pushing, pulling, forcing, accelerating. From these qualitative attributes we get later quantities like energy, momentum, angular momentum and force.

If on the other hand {S} and {O} have been classified already by a set of experiments of type E1 and we are classifying them by another type of experiments E2 it might happen, that both partitions of say {O} are related to each other. In the simplest case both partitions done by E1 and E2 are equivalent. In this cases also the properties of objects relevant for E1 and E2 are related and in the simplest case they can be even the same. It is obvious that this procedure of **relating events** is an extremely important one. It gives another opportunity for the reduction of complexity of the world.

PERCEPTUAL CLASSIFICATION

Some classificatory procedures are so important that one should pay special attention to them. **Observing** is a weak interaction among one subject and a set {O}.

If the observation produces different results they are automatically ascribed to Os. It is namely tacitly assumed that the observers properties relevant for the observation do not change with time. If they do, then we are inclined to say that our senses deceive us. In this interaction of one subject with different elements of {O} we get to the concept of **subjective properties**.

Subjective properties are sometimes difficult to convey to other people. This is where the old Roman's recommendation about the disputes related to taste came from. Subjective perceptual properties are nevertheless very important because they are usually easily accessible.

So it is often the case that **two classificatory** procedures are going on simultaneously: an objective and a subjective perceptual one. If we are rolling objects on an inclined plane which does not change the interaction would result in the classification of all the objects into rollers and nonrollers.

Simultaneously we find out by **looking** to the objects that rollers are seen at least from one side as circles. So we might get to the conclusion: "Circular means roller". It seems that we have found that both these properties are equivalent.

So it comes to us a surprise if we see in the next experiment that some of the cylinders which look exactly the same role and others don't. Subjective operational procedure for finding rollers just by looking at objects is so deeply embedded in our minds that because the cylindrical objects do not role, we imply that they have either some **unobservable** property which explains the difference in their behaviour or that there are some unknown external influences that shape the behaviour of seemingly equivalent objects in different ways. People in the past have been inclined the pack all unobservable properties of a (living) body into soul and all unexpected external influences into god.

Both ideas, soul and god are especially attractive if the classificatory procedure is not repeatable. In this case soul and god are respectively either set of time dependent properties or set of time dependent relations. Time dependency is often expressed by introducing the notion of "will". Will can be ascribed either to soul or to god or to both of them. By the will of individuals and by god's will anything can be "explained". The difficulty of these "explanations" seems to be in their time dependency.

After this analysis it is difficult to subscribe to the theory that functional properties of the objects their temporary aspect which involve change or potential, while the perceptual attributes are permanent (Nelson K, in (7), OU, p. 230).

Of course a cylinder functions as a roller only during the rolling process. But also its redness exists only as long it is illuminated by white light and the observer is looking at it. Time constancy of both properties namely the functional "rollingness" and perceptual readness lies in the repeatability of the relevant experiments.

Why then it seems that children of an early age do not classify objects in accordance to their colour but they do so in accordance to their functional properties?

Functional differences produce different outcomes of classificatory events which can be observed **on the classified objects themselves.** In some cases equivalence classes could be specially dislocated from each other: the rollers are at the bottom, nonrollers at the top of the slope; swimmers are on the surface, nonswimmers at the bottom of liquid; big particles are on the mesh, small ones are under it.

Perceptual differences on the other hand produce different outcomes **in the observer** not on the classified objects. The different outcomes of classificatory process leaves the observer in different sensation and perception states. The classificatory procedure has changed the feelings of the observer, it did not (in general) changed the properties or relations of the observed bodies. The observer has to ascribe his/her feelings to the differences among the observed bodies. This additional mental act is not necessary in the case of functional variables. This might explain the primacy of functional verse perceptual variables in classification procedures. It also emphasises that classification results should be mapped into spatial regions. These could be heaps, drawers, boxes for objects and columns and lines for their names or other symbols.

WHAT CONSTITUTES A CONCEPT

It is possible at this stage to ask oneself if the ability to apply the classification process with respect to one attribute proves that the person who did it has formed a concept of that attribute.

I should be inclined to say "No." It is said that 1 hour old chickens can differentiate a circle from noncircles. But so can water differentiate among substances with different densities. Nobody would say that water has concept of density and it is difficult to claim that chickens have concept of a circle.

This question is often formulated also in terms of procedural verse conceptual (or structural) knowledge. It has been claimed(10) that conceptual knowledge comes as a result of condensation of procedural knowledge. This would imply that the ability (or possibility) to do the classification procedure is necessary but not sufficient condition for having a concept(9).

This view of classificatory behaviour seems to be a bit mechanistic. I am comparing a child with a mash. By this I mean that the sensory basis is prewired in such a way that men can not avoid classifying external stimuli, like a mesh can not avoid classifying stones from sand.

All our sensory apparatus is a **spatial** distribution of receptors for **qualitatively** different stimuli into different regions. Light of different colour are percepted by different receptors in retina, different tastes on different regions of the tongue, there are different receptors for different pitches and different sensors in the skin for temperature and pressure. Qualitatively different stimuli (with different intensive properties) are mapped in perceptual organs onto different spatial patterns.

Quantitative differences among stimuli are represented by the frequency of neuron firing - this means by a **temporal pattern** of transition of the stimuli to the brain.

In the brain both intensity (quality) and extensity (quantity) of stimuli are mapped into a **spatial pattern** of the brain activity although the quality is differentiated mostly by activity in different cortical regions and extensity by the activation of deeper layers of the brains.

I can not see any compelling reason for introducing a higher order explanatory apparatus for the appearance of classificatory behaviour like in this quotation: "The fact that the child groups objects according to discernible principles indicates that these principles are somehow available to him and we assume that they are present in cognition as well as overt behaviour."[(4)]. If however the word "principles" implies also the material structure of the receptors one could comply with this statement.

Interactions among subject and material objects from his surroundings lead to subjective properties of objects which could be named by adjectives or nouns.

However this type of interaction also make changes in the observer. My guess is that these changes might be more perceptible to the subject himself if they are not only weak interactions like looking and listening but of a stronger type where more conscious effort is needed to accomplish it.

The fact that lexical categories of children are **broader** then those of adults could easily be understood if all attributes of bodies and substances are acquired in interactions with one's own body or with some other body. The number of such experiences in young children are small and so only a crude categorisation has been achieved. The naming however has been taken over from adult language which fits much more refined categorisation for which more interactions are needed.

The interactive acquisition of properties also answers the question why only some partitions of the set have been chosen by the child from the vast number of possibilities which grow rapidly with the number of elements in the set. With a set of 10 elements it is possible to make more than 10000 partitions and so it would be possible to introduce a similar number of attributes[(1)].

Classification is necessary condition for the introduction of concepts. Those are introduced to the amount of variability among object into a **smaller** number of categories. So only a negligible fraction of all possible partitions a set should be chosen from a large number of possibilities.

Naming might be the second condition for the introduction of the concept[(9)]. Not every extensively defined class (category) has an operationally defined intensity which is relevant for the subject. Subject makes first only those partition which result from his own interaction with the world and which are relevant for his survival. Only after he can make connections among subjectively and objectively defined attribute, he would broaden his classification interests also to the interactions among different bodies in his surroundings.

This conclusions match with these citations taken from Schyns article:

"Without concepts mental life would be chaotic."
"To think is to forget a difference, to generalise, to abstract."
"Concepts enable an organism to produce similar responses to classes of similar objects, events or situations."
"Concepts reduce an enormous amount of variations into a small
number of categories."

CLASSIFICATION AND CONSERVATION

Sometimes the question is put forward if classification procedure is possible if the classified entities are **not conserved**. This question is a sign of the fact that nonconservative attributes have been overlooked in current research.

Everybody knows that young children **before** the age of 12 months actively engage in the classification of **intensive** properties of substances which they eat. They have to differentiate among temperatures and tastes. They can tell you if they prefer milk or water, warm or cold.

Intensive attributes are exactly those that are conserved when the systems to which they are ascribed are not. In the process of **taking away** temperature and tests are preserved, the volume of milk is not.

Intensive attributes are ascribed to **substances** not to objects. Substances are midway to objects. They lack extensive properties. It could be argued that objects have to have at least **two** attributes.

Substances are conserved if **part** is taken away. Only if **whole** is taken away they are not conserved. By this I mean that their definitional attributes which are intensive are conserved.

If we start classifying objects we miss an important class of entities which are first on the child's classification agenda — substances. A child is a chemists before he becomes a physicist.

Intensive properties are more basic than extensive ones because they are conserved by the activity in which every child engages from the very beginning — this is taking away part from the whole. He is doing that when he is breathing, drinking and eating. From the very beginning he can switch over from of these activities to the other. By this he is able to react differently to gases, liquids and solids.

Eating is a repetitive process for which food has to be divided into chunks by swallowing, biting, gripping first and by flasks, cups and slices next. Some chunks hang around already like apples and peaches. So the concept of body is introduced which has besides intensive also extensive characteristics.

Extension of the body is **not conserved** by eating. It is however conserved if we put the body into the store or if we transform its shape. Extensive properties are good measure for conservation of things but bad measure for consuming them.

On the other hand consumption is the final destiny of everything and in consumption the intensity is the measure of enjoyment, the extensity however is the measure of its duration.

In our society with long waiting time for gratification extensive properties are important, so it is not surprising that we have forgotten about intensive ones. However they do remind us to take notice of them when the quantity is conserved but the stored things become spoiled or rotten.

CLASSIFYING SETS

After the introduction of discrete bodies a set of them can be seen as a discrete model for substance continuum. Through classification a homogeneity is introduced which enable the introduction of **intensive property** of a set which becomes the definitional property of it.

A set of rods can be classified by length or diameter. So even **extensive properties** of bodies are introduced first as intensive properties of sets of such bodies. Extension of a set is measured by its cardinal number obtained counting. This too is obtained as an intensive property of sets of sets.

Counting itself namely does not lead to the concept of cardinality. It is a matching procedure for pairing objects of the counted set with a stipulated sequence of number words. In this sense it is similar to ascribing letters from an alphabet to the elements of the matched set. The letters of the alphabet do not usually acquire any cardinal meaning[2].

If the sections of a book are designated by A, B, C, D we do not expect that somebody would say that the books has D sections.

How exactly the count–cardinal transition is made is not clear. It is however to expect that this transition can be facilitated by presenting children set of sets, for instance set of fivers and some examples of nonfivers. This is in accordance with the previous analysis that extensive properties of object are introduced as intensive properties of sets of such objects. Now the same procedure is applied one step higher: in defining the extensity (cardinality) of a set through intensity of a set of equivalent subsets.

The equivalency of cardinality of the subsets (equipollency) is established by pair matching of their elements.

CLASSIFICATION AND SYMBOLS

Classification procedure based on the equivalence relation leads to two major types of symbolic representations: spacial and verbal. When objects are classified they or their symbolic representations (pictures, signs, words) are put in different spatial regions if they are different and into the same region if they are equivalent. They are given different names if they are different in the attribute which is relevant for the outcome of classificatory interaction. They are described by the same word if they produce the same results in the classification procedure.

By classifying objects we get nouns (names for substances and objects) and adjectives. Through later measurement procedure they will be later described by time **constants** representing intensive and extensive attributes and space variables of continua. Through classification of sets numerals and later numbers are introduced. So we get the whole spectrum of nominal words.

Classifying events yields verbs. Finished events are described by perfective verbal forms which are later described mathematically by state functions and their time integrals. Unfinished events are described by nonperfective verbal forms and adverbs. Adverbs are in mathematics replaced by derivatives. Time derivatives tells us how fast the events progress. Other derivatives (dx/dp, dx/dT, dx/dE) answer more general question how external conditions like pressure, temperature and various fields influence the development of events.

It is important to tell apart changes of intensive attributes and changes of extensive attributes. Time derivatives of intensive attributes are usually called speed, velocity, rate, frequency. Time derivatives of extensive attributes are usually called fluxes, flows and time densities.

It is important to note that our perceptual structure enables us to perceive various fluxes which the body exchange with the surroundings. This are first of all fluxes of matter and energy necessary for the survival. But the senses are perceptible to tiny fluxes and their variations which serve as information carriers. Smell and taste have to do with fluxes of matter, touch, hearing and seeing deal with energy fluxes. It is interesting to notice that in the body all these fluxes are transmitted to the brains in the digitalized form as neuron firing rates.

It is also convenient to speak about events in which discrete bodies take part and separately about events inside large space regions filled with matter or with fields - continua. The description of the events in continuum is possible only after extensive properties of the discrete bodies (mass m, energy E) are replaced by their intensive counterparts which are obtained usually as ratios like m/V, E/V (spatial densities). External influences like forces F are replaced by their space distribution functions like pressure F/S and surface tension F/l which again are intensive. Intensive relations are formed using ratios among spatial differences in intensive properties like temperature T and pressure p and the distances dx. Slopes and spatial derivatives dz/dx, dT/dx, dp/dx are obtained in this way.

Special attention should be given to the classification of relations. The classification itself rest upon equivalence relation. It gives a simple structure to a set because it is universally valid inside an equivalence class. Without such universal validity of some relation in the set we would have just a collection of things. Such collections are predecessors of structured sets. Their members are related to one another by different relations.

Inside education they play a crucial role in various **thematic** approaches or **context** teaching. On the other hand **concept** approaches emphasises the role of structured sets which are the basis for the concept definitions. It has been argued (Sfard A) that **process** approaches do not differ very much from conceptual teaching. It is obvious also from this analysis that the process or procedure which gives rise to equivalence classes (concepts) could be seen as an integral part of the process formation or as the other part of the same coin as Ana Sfard put it. So there might be only two rather different way of teaching: a preconceptual thematic teaching which fits well with lower grades and conceptual teaching which includes also processes leading to the concepts.

ORDERING

Equivalence relation is a powerful tool in science and in every day life. It enables us to generalise our experience with one class member to all members of the same class because the relation is **transitive**. It has however a major deficiency: it does not give any help for the generalisation of experience to the members of different classes. Members of different classes are **not equal** to each other and so the classificatory event will end up with **different** results. But because non-equivalence is not transitive it is impossible to tell anything about what the difference among the outcomes might be.

For this reason it is necessary to try if nonequivalence could be replaced by some other relation which would be transitive. The solution is easy to find in **some** cases. This is **ordering relation** (<, >).

Like classification ordering starts with bodies. It is necessary to find procedures which enable to transform nonequality into > or < relation. This is strict ordering relation. It is interesting to note that ordering is very much out of attention in educational circles nowadays. It is not so in Montessori schools.

M. Montessori has given equal attention to classification and ordering. But her teaching aids are especially designed for strict ordering. This means that each object from a set is different from any other. So her sets are structured only by strict ordering relation.

A more usual case is a set in which some elements are equal to each other so that they can be classified first and after that equivalence classes could be ordered. In this case both classification and ordering are made along the **same variable**. Both procedures could be mapped into space regions. There are also linguistic means to describe ordering especially comparatives and superlatives of adjects and adverbs for ordering two or three elements. If the number of ordered elements is bigger one can described the structure by using ordinal numbers.

It is possible to combine classification along one variable with ordering according to another variable. Of course it is possible to combine ordering along different variables too. Space representations of such ordering are twodimensional or threedimensional arrays of objects or their symbolic representatives. Their final form are histograms and graphs which can be further differentiated into discontinuous and continuous ones. In special cases where each element differs from all the others in the set in two ordinal attributes, double ordering could be represented by onedimensional row. It is interesting to note that M. Montessori has provided educational material for this special case but not for the more general ones.

Ordering enable to introduce **differences** and **ratios**. Only existence of these relations could be shown by ordering. Higher orderings can be based on the comparison of differences for instance. I was not able to find this type of ordering in the literature. It seems however that it is possible to find on the market some teaching aids for children which could be used for this purpose although it is not so sure that they have been designed for it (Ritmo –Nathan, France).

Ordering of events can be done in several different ways. One can compare the changes of definitional variable or the durations of finished events. Both comparisons can be combined into frequencies, velocities and currents which give the possibility to compare also the ongoing events. As mentioned earlier we have good sensorial equipment for this sort of comparisons and ordering.

Through ordering we get to **semiquantitative variables.** They can be either space or time variables. Ordering in accordance to two variables is usually represented by twodimensional graphs. A special case of twodimensional graphs are graphs of **distributions**. Distribution shows a set split into equivalence classes according to some variable which is mapped along one axes. The other coordinate represent the number of elements in each equivalence class. Distribution can be easily generalised also for continuous variables.

The same representation can be used for ordering events in accordance to characteristic change and time duration. The amount of change and the duration of change are represented in **time graphs**. The event is represented by an ordered set of successive values of the characteristic variable. Basic idea for constructing time graphs is common experience that local differences can be transformed into a temporal sequence of perceptions by travelling, that is by space movement. This idea is imbedded in our receptors already because they are about ten times more sensitive for temporal changes than for spatial differences. Time graphs could be read through a narrow slit travelling along temporal axes and perpendicular to it. Time changes are then mapped into virtual movement of a black point. Later we omit the slit as a reading device, our eye just travels along the time axes.

It is also possible to order relations. Spatial and temporal relations are used to define where and when something happened or something exist. This is often done by inserting smaller and smaller space or time intervals into the previous ones. In this way postal addresses are formulated. In a similar way it is possible to define historical or geological periods.

An interesting case are semiquantitative relations among the cardinality of subsets and the set to which they belong. These relations are expressed by words like: nobody, few, majority, all, everybody. This words are semiquantitative predecessors of rational numbers. It is worth to mention that these relations are in the Piaget's work as expressions for intensive attributes. It seems that intensive attributes are defined through these expressions. This meaning of intensity is rather unique.

MEASURING

Counting precedes measuring. Through counting procedure it is possible to ascribe a number to the extension of a set. Immediately after learning how to describe a set with a number, they learn how a number can describe an attribute of a single body.

In ordering procedure each member of the set should be compared with any other. Some of the comparisons can be avoided because of the transitivity of equivalence and ordering relation.

Measurement includes four basic operations: definition of unit, ordering, establishing equivalence and connecting unit elements (concatenation) in such a way that the measured attribute of concatenated system will be additive. Through the operation of concatenation the structure of the set becomes an operational structure which is added to the relational structure given by the equivalence and ordering relation.

The structure of the set with unit and concatenation operation becomes homomorph with real numbers. The results of measurements are real numbers. Special attention should be given to measurements of intensive quantities(5, 6, 8, 11).

MATCHING

Until now I was dealing with two relations only: equivalence and ordering relation. Their fundamental importance is supported by their transitivity which enables generalisations: the equivalence relation inside the equivalence class, ordering relation among the equivalence classes.Transitivity of relations develops mostly from the age of 7 years to 9 years. It depends however on the nature of relation (similarity, equivalence, proportionality, ordering, general matching) and on the nature of the referents (objects, substances,events, sets).In education the influence of the transitive relation has been early recognised through the emphasis on "proportional reasoning". This was later generalised to "relational reasoning" and nowadays it is sometimes called "transitive reasoning". The names does not imply the same concept. However they nicely illustrate the development of understanding what is of crucial importance.Matching one object to another and later elements of one set to elements of another (or the same) set in accordance to some rule which could be expressed as a relation is a very broad generalisation of the activities of classification and ordering. These activities lead to the notion of mathematical **function**.It is possible to generalise also the operations. Until now we have dealt only with the simplest operations of concatenation and addition which both conserve the referent. There are however other operations."Composing two, either like or unlike, quantities to produce a third quantity that is in general like neither of the two original quantities is referred to as referent transforming composition. Multiplication and division are referent transforming composition". (Schwartz J L, p. 41). Some of these operations were mentioned only briefly in bridging the description of interactions among discrete bodies and events in continuum. They are important means for the reduction of number of variables. However they are only rarely directly perceptible and so are more abstract. Reduction of complexity can be bought only by increasing abstractness.

1 Ahn W, Medin D L, A two-stage model of category construction, Cognitive Science 16, 1, 1992 p. 81-121

2 Fuson K C, Hall J W, The acquisition of Early Number Word Meanings. A Conceptual Analysis and Review. in Ginsburg H P (Ed.), The development of mathematical thinking, Academic Press, London, 1983 p. 49-107

3 Gordon C.K., Introduction to Mathematical Structures, Dickerson Publ. Co, Belmont, 1967, p. 47

4 Johnson–Laird P N, Watson P C (Eds.), Thinking, Reading in Cognitive Science, Cambridge University Press, 1980 (OU Set Book)

5 Narens L, Abstract measurement theory, MIT Press, Cambridge, Massachusetts, 1985

6 Nelson D, Measurement in School Mathematics, National Council of Teachers of Mathematics, Virginia 1976

7 Nelson K, Some evidence for the cognitive primacy of categorisation and its functional basis, In P N Johnson Laird, P C Watson, Eds, Thinking, Readings in Cognitive Sciences, Cambridge University Press, Cambridge 1980

8 Schwartz J L, Intensive Quantity and Referent Transforming Arithmetic Operations in Hiebert J, Behr M (Eds.) Concepts and Operations in the Middle Grades, The National Council of Teachers of Mathematics, LEA, 1989

9 Schyns P G, A Modular Neural Network Model of Concept Acquisition, Cognitive Science, 15, 4 p. 461 - 508

10 Sfard A, On the dual nature of mathematical conceptions, Reflections on porcesses and objects as different sides of the same coin, Educational Studies in Mathematics, 22 (1991), p. 1-36

11 Vergnaud G, Multiplicative Structures in Lesh R, Landau M (Eds.) Acquisition of Mathematics Concepts and Processes, Academic Press, New York, 1983