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Modifying Elementary School Teachers' Conceptions of Mathematics and Mathematics Teaching and Learning : A Strategy Based on Conceptual Analysis

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PROBLEM AND HYPOTHESIS

Four problems, tied to the nature of mathematics, to the philosophy of mathematics, to epistemology and to learning theories, plagued mathematics education for years. And they still do. These problems are :

- the heavy emphasis put on symbolism and notation (Ginsburg, 1977),
- the great influence of formalism (Davis et Hersh, 1980),
- the heavy presence of behaviorist learning theories (Dionne, 1988),
- the exagerated focus that many teachers put on their pupil's answers instead of on their reasoning (Dionne, 1988).

As these problems persisted through both time and reforms, we must look elsewhere than in curriculum transformations for their solution. As we have already said, these problems are tied to philosophic perceptions of malthematics and to conceptions of learning. Thus we should address the latter if we want to remedy the situation and improve mathematics teaching. And in this undertaking, teachers are the targets we should favour : firstly, because they are the true workers of mathematics teaching and thus, they are the pivoting factor of any genuine renewal of that teaching (Colmez, 1979; Dionne, 1983, 1988). But also because they have been neglected in all past reforms in the sense that they seldom received the necessary profiency courses and that their pre-service training was often barely adequate (Freudenthal, 1977; Beltzner et al., 1977; Colmez, 1979; Robitaille and Dirks, 1982; Dionne, 1983, 1988). A new approach of teaching training, whether pre or in-service, seems to be a sine qua non condition to any possible solution of the problems at hand.

Such a new approach was test in our research. As already described in Bergeron et al. (1981), Herscovics et al. (1981) and Dionne (1988), this approach is based on the initiation of teachers to the analysis of mathematics concepts within the framework of a model of understanding. More precisely, we avoid to study mathematical concepts or notions from a strictly mathematical point of vue, asking or answering questions like : "What is a number?" We also avoid to consider general interrogations on understanding such as : "What is understanding?...". Instead, we put these elements together into more practical and answerable questions like the following : "What does it mean to understand the notion of number?" Such questions are brought within a frame of reference based on models of understanding, which allow the student teachers to explore the concept in a meaningfull way. We present one of these models in the next section. But let us say right now that these models do not attemp to give a definition of understanding but they try to identify appropriate criteria for its description. Of course, as explain Herscovics and Bergeron (1983), these criteria provide the means for a cognitive analysis of the various concepts taught, as well as an interpretation of what is meant by understanding in a wider sense than just "giving the right answer". In such models, understanding is not seen as an instantaneous event, but is really presented as a cognitive construction process.

This approach for teacher training in mathematics links the psychological, epistemological and pedagogical aspects of the teaching and learning of mathematics to the mathematical content itself. Mathematical concepts being examined in the context of the model of understanding, the focus thus put on the cognitive aspects leads the teacher to reflect on the mental processes involved in the elaboration of a particular notion. This, in turn, may also bring the teacher to restructure his or her teaching strategies in order to conform more closely with what he or she has understood about the learning of the notion. According to our hypotheses, **this approach should lead the teachers to a more constructivist perception of mathematics and of mathematics teaching and learning**.

THE CONSTRUCTIVIST MODEL OF UNDERSTANDING USED IN THE RESEARCH

The model of understanding which was used in our research is called **The Constructivist Model of Understanding** because, as already said, it presents understanding as a **cognitive construction process**. This model was developped by Herscovics and Bergeron (1983) from previous models suggested by Bruner (1960), Skemp (1976), Byers et Herscovics (1977), Herscovics and Bergeron (1983). It identifies four levels of understanding organised linearly as in the figure below : the first one is called intuition, the second one involves procedures, the third one deals with abstraction and the fourth and last one, with formalization.

Constructivist Model of Understanding

Intuitive Understanding	Procedural Understanding	Abstraction	Formalization

Research on the acquisition of fundamental concepts has showed that, even at elementary levels, their construction stretches out over a period of many years (Piaget and Szeminska, 1947/67). That's why understanding can hardly be considered as an event, but should be looked at as a process in which one can find many ways to understand a given notion : this is what is emphasized in this model.

Intuition or Intuitive Understanding

For most of the concepts studied in elementary school, one can find preconcepts which can be viewed as embryonic to the conceptual schema whose construction is intended. For instance, the concept of cardinal number is based on the notion of quantity : this preconcept is developped well before any schooling and is based initially on sole (mainly visual) perception, without any counting action. This is enough though to enable the child to judge if quantities are the same, more or less, and to distinguish between few and many. Of course, such visual estimation give rise to errors due to limitations of visual perception which is affected by the configuration of a set or the sizes of objets to be compared.

This illustrates what is meant by intuitive understanding : this level of understanding is characterized by informal knowledge, preconcepts, a type of thinking based on visual perception and primitive unquantified actions limited to rough approximations.

As Herscovics and Bergeron (1983) explains, there are many reasons psychological as well as pedagogical, to select this kind of informal mathematics as a first level of understanding. From a pedagogical point of view, this stands induces a search of appropriate situations from the child experience, providing the acquisition of new knowledge with meaning and relevance, and thus leading to an increased motivation. From a cognitive point of view, making sure that the child's mathematics are solidly rooted in his world and experience, allows this child to construct his conceptual schemata without cognitive discontinuity.

Procedural Understanding

Of course, this kind of informal mathematics does not suffice : the child who is asked to choose between two piles of 18 and 20 "smarties" respectively, will soon develop some procedure in order to select the "best" one, according to his or her degree of greediness... The first procedures are generaly physical : for instance, the child will use a 1-1 correspondance. Later, he or she will be able to use counting actions, thus using much more sophisticated and really mathematical procedures.

The assimilation of such procedures, which usually results from pedagogical interventions and render his or her knowledge more precise and operational, constitutes the second level of cognition of the model, the level of procedural understanding. This level is evidenced by the acquisition of an initial procedure which coordinates intuitive knowledge and some prerequisites (the ability to recite the number word sequence for instance) to provide the means for a systematization (e.g. quantification, organization).

Abstraction

Procedures as the one described above bring about the formal construction of mathematical notions or objects such as number, or mathematical transformations such as arithmetic operations. But in the first stages, the intended concept is often blurred and confused with the procedure leading to its construction : it's only gradually, for example, that the concept of cardinal number separates from the counting procedures and starts to have an existence of his own in the child mind. This separation is a first step in the process of abstraction.

As time goes, the child becomes aware of the invariance of number with respect to the configuration of a set of objects, or with respect to the order in which these objects are counted. The process of construction of such invariants characterizes a more advanced phase of mathematical activity.

These observations lead us to the following description of abstraction : abstraction is initially characterized by the separation of the concept from the procedure and then by its generalization, or by its conservation which reflects the invariance of the mathematical object. To this, one can add the reversibility and composition of mathematical tranformations.

Formalization

Up to now, there were no mention of the mathematical symbolization, even if the particularly symbolic nature of mathematics is well known. This stresses the fact that symbolization is much more an end product of learning —a way of expressing what has been learned — than a way of learning. (We should bring some nuances here, but you can bring them by yourselves, we guess). On the other hand, everybody has met students who could succeed on tests by manipulating symbols which were devoid of any meaning to them and by basing themselves solely on the disposition of the symbols to derive idiosyncratic rules (Erlwanger, 1973, 75 for instances). This explains why, if symbolization is considered as relevant in the description of the fourth level of understanding, it is considered so only if prior abstraction of the concept has occured to some degree.

Thus, depending on the notion intended, formalization is characterized either by the use of symbolism, or, as often interpreted in mathematics, by the logical justification of operations or the discovery of axioms. But in all cases, this level presumes prior abstraction.

Back to the main hypothesis of the research

We have said that this kind of model allows the pre or in-service teacher to explore mathematical concepts in a meaningfull way. This way is meaningfull indeed, not only because the mathematical concept remains present in what is submitted to the teachers' reflection, but also because these teachers's mathematical reflections are embedded in psychological and pedagogical concerns. Thus, the focus is put on the way children build their mathematical knowledge. And this shows explicitely that mathematics do not suddenly break out in the mind of people, but are the result of a process of construction. These are the reasons why we formulated our hypothesis : our new approach for teacher training, based on concept analysis with the help of a model of understanding, should lead the teachers to a more constructivist perception of mathematics and of mathematics teaching and learning.

EXPERIMENTATION AND TOOLS

We tested our hypothesis on an experimental group (n=18) of teachers enrolled in an in-service program at the "Faculté d'Éducation Permanente" (F.E.P.) at the Université de Montréal. These teachers took a 45 hour mathematics education course in which our approach was used. Simoultaneously, we appealed to a control group (n=16) of teachers also enrolled in the same F.E.P. program but taking a course in a field other than mathematics or mathematics education. Immediatly before and after the 45 hours courses followed by both groups, the participating teachers were subjected to a test and a questionnaire. Furthermore, six participants from each group accepted to answer twice the same long interview, once before and once after the course. Thus, it became possible to evaluate the perceptions shown by the teachers and to evaluate as well the evolution of these perceptions. Therefore, three tools were developed, a test, a questionnaire and an interview : these three tools are described in the following paragraphs.

The "Correction Test"

In the teaching of mathematics, because of an instrumental and formalist perception of mathematics as well as a rather behaviorist vision of the learning process, teachers often attach more importance to the mathematical products than to the mathematical processes, often reducing understanding to the acquisition of abilities. This is reflected by a tendency to focus more on the student's answer than on his or her reasoning. With the new approach, as said, we thought that the focus put on the cognitive aspects of teaching would lead the teacher to reflect on the mental processes involved in the elaboration of a particular notion. This, in turn, may bring him or her to restructure his or her teaching and evaluation strategies to conform more closely with what he or she has understood about the learning of the notion. So, the Test was elaborated to evaluate the relative importance the teacher places on the students' answers and on their reasoning.

The test is composed of two elementary problems, one in geometry (What is the area of a rectangular garden whose sides are 3 m and 6 m respectively?) and the other one in arithmetic (Peter has four dozen eggs. Jack has only two dozen. How many more eggs does Peter have?). Each problem was given to teachers along with four possible student solutions. The solutions were of the following four types.

Type 1: a right answer stemming from a correct reasoning.
Type 2: a right answer by itself, without any trace of reasoning.
Type 3: a right answer but originating from a faulty reasoning.
Type 4: a wrong answer in spite of a good reasoning.

Teachers were asked to grade (on 10) each solution and then to justify their grading. We thought that a teacher who assigned more importance to the answer than to the reasoning would give low marks when the answer is wrong, even if the reasoning is correct (type 4) and high marks when the answer is right, despite a faulty reasoning (type 3) or no evidence of reasoning (type 2). On the contrary, a teacher more interested in reasoning would give high marks when the reasoning is correct even if the answer is wrong (type 4) and lower marks when the reasoning is incorrect (type 3) or missing (type 2). The justifications were required in order to validate our interpretation of the marking.

The "Questionnaire sur la Perception des Mathématiques"

If we consider the "world of mathematics", we might perceive it in different ways; three of these perceptions seem particularly important if one thinks about school mathematics :

- the traditional perception, which is found among a majority of people, usually those who are not in the field of mathematics. Mathematics is seen from that perspective as a collection of rules, these rules being considered as tools which give the user some power over reality. But this power is somewhat mysterious and magic, and thus a bit frightening. People feel that they cannot control it because, while they see the results it brings, they don't understand how and why it works. In the schools, this traditional perception appears when mathematics is limited to rules and calculations i.e. when the "skills" component dominates if not eradicates the other components of real understanding ;

- the formalist perception which comes out of what has been, for a long time, the dominant philosophy in mathematics; ultimately, mathematics is seen from this perspective as the science of formal structures and of rigourous logic. The mathematical content, the meaning are relegated to what is called meta-mathematics. In schools, this perception appeared, mainly during the "modern math" era, in the emphasis of unifying mathematics around a few basic concepts (sets, function…) and on the search for rigour in langage and symbolisation. The early insistence of these formal aspects has often contributed to their lack of meaning for many people ;

- the constructivist perception, not in the mathematical sense but in the epistemological one. According to this perspective, one gets a piece of knowledge if and only if one has reconstructed it for oneself : thus, knowledge is no longer an object to be transmitted to but an object to be built by the learner. If one wants an efficient teaching of mathematics, one should see mathematics as a field of knowledge similar to other fields of knowledge. And one must agree, as Revuz (1980) says, that for the child who is learning as well as for the mathematician who is creating it, mathematics is something to do and not only to look at. Thus, we must give importance to intuition and to all elements which play a part in the building of real understanding.

The tool developped to evaluate the presence and relative importance of these three perceptions, called "Questionnaire sur la Perception des Mathématiques", was also used to evaluate changes in teachers' perception of mathematics. The "Questionnaire" proposes three sets of statements, each set characterizing one of the perception already described. These sets are :

A- Mathematics seen as a set of skills (traditional perception) :

Doing mathematics is
- doing calculations ;
- using rules ;
- using procedures ;
- using formulas ;

B- Mathematics seen as logic and rigour (formalist perception) :

Doing mathematics is
- writing rigourous proofs ;

- using a precise and rigourous langage ;
- using unifying concepts ;

C- Mathematics seen as a constructive process (constructivist perc.) :

Doing mathematics is
- developping thinking processes ;
- building rules and formulas from experiences on reality ;
- finding relations between different notions.

People are first asked to order these sets according to the importance they assign to each one, using the symbols = or > : for instance, one can answer C = A > B. Then, they have to quantify that classification, by giving marks (for a total of 30) to the three sets of statements : the highest score is to be given to the set that is classified as the most important and the lowest to the least important one. For example, one can give 12 each to A and C and 6 to B.

The interview

The third tool was a long interview in which were explored the teachers' perceptions of mathematics teaching and learning. Four themes were addressed : the role and place of intuition, the role and the place of understanding and of skills, the part of discovery and of definitions and, finally, the role and importance given to errors. Within each part of the interview, teachers were asked to specify the meaning they gave to key-words used, to illustrate that meaning with examples and to articulate the role they attribute to each of these elements in their class practice.

THE MODES OF ANALYSIS

As it was already said, the first two tools, the test and the questionnaire, were used with each group, experimental and control, each individually taken as a whole. Results concerning procedures and conclusions are presented below. After what. we concentrated on the twelve case studies related to the interviews. But in each of these twelve cases, the response to all three instruments (test, questionnaire and interviews) were considered and examined as a whole. The purpose of this approach was to clearly isolate the manifest perceptions and their evolution. Next, the answers to "pre" and "post" interviews were analysed. These analyses were afterwards compared to those conducted by two independant experts. The only conclusions retained from this exercise were those on which agreement was unanimous. What should be added here is that these analyses were more qualitative than quantitative. They were based on the search for coverging indications : this appears to be more interesting since we do get a real description of the perceptions instead of mere reductionist statistics. And the search for convergent indications - in which we tried to validate the answer to the test, to the questionnaire and to the different parts of the interview - allows for clearer conclusions on a topic which in itself remains difficult to define and delimit.

RESULTS OBTAINED FROM THE "CORRECTION TEST"

Type 1 solutions

Type 1 solutions are perfect : they show right answers stemming from clear correct reasoning. for this solution, our aim was not only to detect any change in the marks given before and after the courses, such a change reflecting inconsistency on the part of the teachers being tested, but to make sure that these teachers understood the problems and were able to grasp the soundness of the reasoning and the accuracy of answers.

On the geometry problem, 15 out of the 16 teachers in the **control group** gave 10/10 in the pre and the post-tests . In arithmetic, 15 teachers awarded 10/10 in the pre-test and all 16 gave that perfect score in the post-test. This was a good first indication of consistency on the part of the teachers. Teacher no 4 was the only exception : this teacher's justifications revealed a perfectionnist nature which led her to deduct marks for minor formal details while recognizing the overall quality of the solution.

Marks given Type 1 - solution Right answer, Good reasoning			
Geometry		Arithmetic	
Control	Experim.	Control	Experim.
Pre Post	Pre Post	Pre Post	Pre Post
10 - 10	10 - 8	10 - 10	10 - 10
10 - 10	10 - 10	10 - 10	10 - 10
10 - 10	9 - 10	10 - 10	10 - 10
8 - 7	10 - 10	9 - 10	10 - 10
10 - 10	10 - 10	10 - 10	10 - 10
10 - 10	10 - 10	10 - 10	10 - 10
10 - 10	10 - 10	10 - 10	10 - 10
10 - 10	10 - 10	10 - 10	10 - 10
10 - 10	9 - 10	10 - 10	10 - 10
10 - 10	8 - 10	10 - 10	10 - 10
10 - 10	10 - 9	10 - 10	10 - 10
10 - 10	10 - 10	10 - 10	10 - 10
10 - 10	5 - 10	10 - 10	10 - 10
10 - 10	8 - 8	10 - 10	10 - 10
10 - 10	10 - 10	10 - 10	10 - 10
10 - 10	10 - 10	10 - 10	10 - 10
	10 - 10		10 - 10
	10 - 10		10 - 10

In the **experimental group**, perfect scores were awarded by all teachers for the arithmetic problem on both the pre and post-tests while in geometry, only 11 of 18 teachers gave perfect marks in the two cases. Though the latter was at first alarming, the teachers' comments showed that 17 out of the 18 teachers acknowledged the correctness of the reasoning and the answer in the pre-test, and that all 18 acknowledged this in the post-test. The slight variations observed in the grades reflect a more stringent attention to details in the presentation of the solution. The only real discrepancy was teacher no. 13 in the pre-test, whose reaction can be explained by the fact that she looked at the type 3 solution (right answerbut faulty reasoning) immediatly prior to evaluating type 1 and was thus biased.

The preceding demonstrates that :

- all the teachers faced the task of grading in a responsible and competent manner;
- both groups seemed consistent in their grading though teachers in the experimental group proved to be more demanding with respect to form. Fortunately, this tendency can easily be monitored by their own justifications;
- the two groups seemed similar enough to be used in further comparaisons.

Type 2 solutions

Type 2 solutions are the shortest; in the arithmetic problem, all we find on the page is the answer, 24 eggs; the geometry question is much the same with the addition of a simple drawing to the answer, 18 m^2. There is absolutely no trace of the reasoning involved present on either answer sheet. We predicted that a teacher for whom the answer is the most important would grade these answers generously while, rather, a teacher who is more interested in the reasoning would give low marks unless he was to give the student the benefit of doubt. Thus, a shift of focus from the answer to the reasoning would be reflected by a drop in the marks awarded.

This, however, is not what was observed. Both groups behaved similarly in the pre and post-tests, the marks given follow no distinct trend :

- in the **control group**, among the 14 teachers who did complete the test for this type of solution, 4 awarded the same mark before and after in geometry while 3 of them increased the grade and 7 lowered it. In arithmetic, there were 9 the same, 2 increases and 3 decreases;
- in the **experimental group**, out of 18 teachers in geometry, 4 stayed the same, 7 decreased and 7 increased their marks while in arithmetic, 7 went unchanged, 5 increaed and 5 decreased, 1 of the original 18 teachers not completing the post-test.

Marks given type 2 - solution Right answer, no reasoning			
Geometry		Arithmetic	
Control	Experimental	Control	Experimental
Pre - Post	Pre - Post	Pre - Post	Pre - Post
10 - 10	6 - 10	10 - 10	6 - ?
5 - 0	3 - 2	1 - 0	3 - 2
5 - 5	5 - 7	5 - 5	3 - 6
5 - 2	0 - 3	0 - 0	0 - 0
5 - 10	0 - 0	5 - 10	0 - 0
10 - 8	8 - 3	10 - 4	0 - 1
7 - 3	10 - 10	1 - 1	10 - 10
3 - ?	4 - 0	3 - 3	4 - 0
5 - 0	7 - 10	0 - 0	5 - 2
5 - 8	4 - 4	2 - 2	2 - 2
0 - 2	10 - 9	0 - 0	10 - 8
4 - ?	2 - 0	4 - ?	2 - 0
5 - 2,5	5 - 7	0 - 2,5	5 - 6
2 - 2	5 - 8	2 - 2	5 - 5
5 - 5	5 - 1	5 - ?	0 - 1
8 - 5	5 - 5	5 - 3	5 - 5
	5 - 8		5 - 5
	10 - 9		0 - 10

No definite conclusions can be drawn from the type 2 solutions since the variations seemed inconsistent. The only possible observation is that a large majority of teachers in both groups commented on the absence of detailed answers and stated that they would not accept this kind of solution from their own students.

Type 3 solutions

Type 3 solutions are the most disturbing : correct answers are presented, stemming from a faulty reasoning (the perimeter is calculated instead of the area in geometry) or from a fragmented reasoning (only the number of eggs that Jack has is calculated in arithmetic). Once again, we expected lower grades from the teachers who are more concerned with reasoning and higher grades from the ones concerned with the answer, and a shift of focus from the answer to the reasoning being reflected by a drop in marks awarded.

Marks given type 3 - solution Right answer, faulty reasoning			
Geometry		Arithmetic	
Control	Experimental	Control	Experimental
Pre - Post	Pre - Post	Pre - Post	Pre - Post
0 - 0	3 - 10	0 - 0	5 - 10
0 - 0	3 - 5	5 - 6	3 - 3
2 - 3	10 - 0	5 - 5	4 - 4
2 - 0	0 - 0	3 - 3	5 - 4
0 - 0	0 - 0	0 - 0	0 - 0
2 - 4	10 - 0	3 - 8	0 - 0
5 - 0	0 - 10	3 - 3	10 - 10
0 - 0	0 - 0	8 - 5	5 - 5
10 - 10	9 - 10	3 - 3	0 - 2
0 - 0	4 - 4	3 - 5	2,5 - 10
0 - 2	10 - 8	4 - 5	0 - 9
0 - 0	2 - 1	8 - 3	5 - 5
0 - 10	5 - 5	0 - 0	7 - 7
2 - 2	1 - 7	6 - 6	6 - 7
0 - 10	2 - 0	3 - 10	3 - 3
? - 0	0 - 5	5 - 5	7 - 8
	0 - 8		5 - 3
	10 - 0		0 - 0

But once again, our observations differed from our expectations:

- in the **control group**, 8 of the 15 marks awarded in geometry remained the same in the pre and post-test, 5 increased and 2 decreased; in arithmetic, 9 marks out of 16 remained unchanged in the post-test while 5 increased and 2 decreased;

- in the **experimental group**, 5 marks out of 18 were unchanged for the geometry problem, 7 marks increased and 6 decreased; in arithmetic, 10 marks stayed the same, 6 increased and 2 decreased.

There is no clear trend leading to a definite conclusion in these variations since both in the experimental and control groups, almost equal numbers of marks increased in the arithmetic problem as well as in the geometry problem. Exactly the same number of marks decreased in the arithmetic problem in both groups; the only exception was the geometry problem where there were 2 decreases in the control group contrasting with 6 decreases in the experimental group. Moreover, about one third of the teachers in both groups did not detect the mistakes in the reasoning in the pre or in the post-test or in both cases, although every teacher commented on these reasonings in his justifications : some of the teachers found them correct, some said that they were faulty, the others, disturbed by the exactness of the answers, did not know how to deal with these faulty reasonings and seemed to think that “something” ought to be good “somewhere”....

Type 4 solutions

Type 4 solutions are almost perfect : they show correct reasoning but wrong answers because of poor calculations (6x3=16 in the geometry problem) or mistaken transcription

Marks given type 4 - solution Wrong answer, good reasoning			
Geometry		Arithmetic	
Control	Experimental	Control	Experimental
Pre - Post	Pre - Post	Pre - Post	Pre - Post
5 - 8	7 - 8	8 - 8	8 - 8
8 - 8	7 - 8	8 - 8	7 - 9
8 - 8	7 - 9	7 - 8	8 - 9
7 - 6	7 - 8	6 - 6	7 - 8
8 - 9	8 - 8	9 - 9	8 - 8
8 - 9	8 - 8	9 - 9	8 - 6
7 - 7	0 - 0	9 - 9	0 - 0
8 - 8	6 - 8	8 - 8	8 - 8
8 - 8	4 - 8	8 - 9	7 - 8
8 - 8	5 - 7	7 - 8	8 - 8
8 - 8	8 - 8	8 - 8	8 - 7
7 - 7	8 - 8	7 - 7	8 - 8
0 - 7,5	5 - 8	5 - 10	7 - 8
8 - 8	7 - 7	8 - 8	5 - 6
0 - 0	8 - 9	6 - 6	8 - 9
5 - 5	7 - 8	5 - 5	7 - 8
	5 - 8		5 - 8
	5 - 8		8 - 8

of number (42 instead of 48 in the arithmetic problem). We predicted that a teacher more concerned with the answer would give low marks for these solutions and a teacher more interested in the reasoning would award higher marks, a shift of focus from the answer to the reasoning being reflected by an increase of the marks awarded.

This time, we observed a marked difference between the experimental group and the control group :

- in the **control group**, 11 teachers out of 16 gave exactly the same marks in the pre and post-tests for the geometry problem and 12 of them did the same for the arithmetic problem. Moreover, 4 of the 5 remaining marks changed for the geometry problem and the 4 remaining marks for the arithmetic problem increased. This could be interpreted as a "natural" trend to increase the marks;
- in the **experimental group**, 12 teachers out of 18 increased their marks in the post-test for the geometry problem while 6 other marks remained unchanged. For the arithmetic problem, 9 marks were increased, 2 were decreased, the remaining 7 staying the same in the pre and post-test.

This seems to confirm our hypothesis that the course in conceptual analysis would bring about a change in the way teachers weighted the reasoning and the answer. Our results indicate that, while 4 out of 16, 25%, of the teachers in the control group "naturally" increased their grading from the pre to the post-test for the geometry problem, this percentage climbs to a striking 67% (12 out of 18) in the experimental group. This trend is strongly confirmed by what is observed for the arithmetic problem : the same 25% upward shift is present in the control group, while 50% (9 out of 18) of the teachers in the experimental group awarded a higher mark in the post-test than they did in the pre-test for this solution.

Conclusion on the "Correction Test"

The focus the teachers put on the student's answer or on his reasoning remains hard to mesure because it is a **tendency** and it is not as clearcut as one may think. No teacher would consider only one of the two elements, answer or reasoning, and completely exclude the other from his preoccupations : this is what type 2 solutions (none of the teachers would accepts an answer alone) and type 3 solutions (where a good answer proved to be disturbing for a large nmber of teachers) showed. Thus, the variations observed in the marks given were often thinner than one may expect.

However, type 4 solutions clearly indicate that a change occured in our experimental group : teachers initiated to conceptual analysis proved to be more concerned with the reasoning of the child in the post-test than they were in the pre-test. A change in the same direction was also observed in the control group but was, by far, less important.

RESULTS OBTAINED FROM THE "QUESTIONNAIRE SUR LA PERCEPTION DES MATHÉMATIQUES"

The following table presents a résumé of the results obtained :

Mean of the marks given to each set of statements

Perception	A (traditional)		B (formalist)			
Group	Pre	Post	Pre	Post	Pre	Post
Control	7,5	10,4	9,5	9,8	13,0	9,8
(n = 15)	+2,9		+0,3		-3,2	
Experimental	10,8	9,5	6,3	7,5	13,7	13,3
(n = 18)	-1,3		+1,2		-0,4	

This table is based on the marks given and neglects the classification : the reason for this is that for nearly half the teachers, there was a contradiction between their first classification and the marks given afterwards. For example, on one answer sheet we had C > B > A and C = 10, B = 15 and A = 5. Fortunately, a large number of teachers had made remarks showing that the marks given were the thing to consider as they were not very familiar with the use of the symbols > and = when more than two variables were to be classified. Only one person — in the control group — was rejected because her answer in the postest was impossible to interpret.

Both groups have changed between the pre and the postest. This is not very surprising since most teachers were confronted to the question of the nature of mathematics for the first time in their life, and the pretest had the effect of starting them thinking on the topic and eventually changing their view. However the change was not at all the same for the two groups.

In the pretest, both groups have quite similar means for set C to which both give the most marks, but they differ for sets A and B, the control group giving more importance to rigour (B), and the experimental group to skills (A).

In the postest, the control group has clearly evolved towards a balance among the three perceptions, all of which were given almost the same importance (~10) : this is not surprising since in the sets characterizing each perception, there are statements whose importance cannot be denied.

On the other hand, the experimental group remains clearly convinced of the dominant importance of set C while tending to equilibrate the marks given to sets A and B. This may be a direct consequence of the course followed by the teachers in this experimental group. The focus in that course was on the way children build their understanding of mathematics, thus reinforcing indirectly a constructivist conviction which was less firmly held by the teachers in the control group. The latter, after a period of thinking between the pre and postest, were not able to give an hierarchy.

Conclusion on the "Questionnaire sur la perception des mathématiques"

To sum up, two elements seem important :

- the tool described here allows an evaluation of the different perceptions of mathematics held by teachers ;
- however, these perceptions are not as clearcut as one may think : a teacher who prefers one perception will still consider important some elements of the other ones. This explains why evolution in this area is slow and remains hard to mesure.

RESULTS OBTAINED FROM THE TWELVE CASE STUDIES

The six teachers in the control group

Of the six subjects involved with the control group, five showed constructivist perceptions and tendancies while the sixth one, Lise, appeared much more traditional. As all the perceptions did not change much between the pre and post-tests, the descriptions that follow in the next paragraphs hold for the period that preceeds as well as for the period that follows the proficiency course followed by these subjects.

In their reactions to the correction test, the six subjects proved to be much more concerned with the reasoning of the children than with their answers. There were only few variations in the marks given when, for each category of children's answers, we compare the geometry problem to the arithmetic problem or the pre to the post-test.

It is much more hazardous to try to conclude from the individual answers to the questionnaire : there was often a contradiction between the first classification given by some teachers and the marks they gave afterwards. Global analysis based on the groups' means showed interesting indications. However, it is much more difficult, if not impossible, to characterize in a few general statements the individual reactions of the subjects.

Intuition, the first theme of the interview, was first described in general terms ("to guess", "to anticipate") but four of the six teachers succeed in linking this concept of intuition to learning. For example, they spoke of instinct, trial and error discovery, use of concrete materials. Later, while addressing the theme of learning by discovery, three teachers stood up in defense of very intuitive approaches based on manipulation of concrete objects, but they did not use the words "intuition" of "intuitive" to characterize these approaches.

Most of the teachers acknowledged the importance of understanding. They described this notion in terms of explanations given by the child, two of them adding the idea to transfer. Lise distinguished herself as she linked the idea of understanding with the ability to memorize and to apply or to use. The idea of skill, also dealt with in this second part of the interview, remained vague for most of the subjects. Only two of them were able to define it in terms of "ease at doing something".

Five out of the six subjects agreed on the effectiveness of learning by discovery. They prefered this approach to what is known as the traditional magisterial approach in which the teacher defines the notions to be learned. Lise, even if she thought of discovery us an "ideal" way of learning, admitted that she confined herself to traditional, authoritative lectures on mathematical concepts and operations.

Finally, mistakes were perceived as normal and even useful by all the six participants. They did not all believe that a good answer can stem out of bad reasoning. However, if such and event were to occur, they thought that the student's explorations would allow them to realize what had happened. Only one teacher claimed that she systematically asked the children for explanations...

With these people from the control group, as we have already pointed out, changes were rare and isolated phenomena, so that, as such, they never indicated to any substantial evolution in these subjects' perceptions. Hence our conclusion that our tools did not initiate modifications of these perceptions.

The six teachers in the experimental group

In both the pre-test and the pre-interview, our six experimental subjects proved to be very much similar to those of the control group : five of them appeared to be rather constructivists, showing reactions analoguous to the ones already described. Furthermore, Jacques, like Lise, proved to be more traditional. However, changes observed between the pre and post-tests or between the pre and post-interviews were much more numerous and coherent. The next paragraphs give a description of these.

In the correction post-test, our first subject pays a much more nuanced attention to the child's thought processes as she distinguishes between a computation mistake and a reasoning error. And this is confirmed, in the post-interview, by the clear distinction she establishes between poor undertanding and a lack of skill. Our subject then describes what she considers her "new" reactions when facing students' mistakes, explicitly asserting here the influence of the experimental course. She also claims this influence when she explains, still in the post-interview, that she now distinguishes levels of understanding and can describe their manifestations.

In the correction post-test, our second subject deprived her students of the benefit of the doubt she had given them in the pre-test. This increased severity is explained in the post-interview when our subject relates how much surprised she was when, asking questions to "good" students, she discovered that they simply could not explain their solutions to particular problems. Still in the post-interview, she presented intuition as a kind of spontaneous and primitive knowledge and described different levels of understanding. For instance, she mentioned that these levels allowed the teacher to overcome the simplistic "you have it or you don't" diagnosis. Finally, she insisted on the ability to explain as an important criterion of understanding, a criterion replacing the ability of problem-solving. This attitudinal change is coherent with the increased severity mentioned above.

Of all the interviewed persons, our third experimental subject was the most constructivist from the outset. Thus, she did not change much but even then, she deepened and structured her beliefs. The course allowed her, notably, to clarify her perceptions of intuition, now seen as the first step in the understanding process and to describe that intuition in terms of cognition and not only in terms of representation. She now could also establish a classification of errors distinguishing between a technical mistake, a wrong choice of process and a lack of understanding.

Our fourth subject was also a very constructivist person but this did not prevent her from renewing her vision of intuition which she described, in the pre-interview, as a vague feeling and, in the post-interview, as primitive knowledge, an important step in the learning process. She also asserted, in the post-interview, that she now believed more than ever in the importance of the process of discovery. And she associated the latter with the manipulation of concrete materials, which, as we believe, constitutes a rather intuitive approach.

Our fifth subject also showed signs of change in her perception. For instance, she explained how the course helped her to get a better understanding of basic arithmetic and, thus, a better understanding of the roots of her students' difficulties. And as she acknowledged this, she was led to a more general use of concrete materials and manipulations, and for her, this was added proof of the importance of intuition and discovery in the learning process.

Jacques, our last subject, was at the beginning what he remained throughout the experiment : a very traditional teacher. For instance, in the correction test, he remained, above all, interested in answers and did not see any place for intuition in the learning process. He did not find "realistics" the use of discovery learning nor did he consider error as something rather difficult to analyse. His conviction were, for sure, a little more "nuancées" than what this brief summary makes then to be, but as it may, they remained unaffected by the experimental course.

Conclusion on the case studies

There is a clear convergence of indications : five out of the six control subjects proved to be rather constructivist. For instance, their reflections on errors were coherent with their reactions in the correction test; the attention they showed, in that test, for the student's reasoning process was confirmed by the importance they attached to understanding. Similarly, the importance they gave to intuition was confirmed by their marked preference for learning by discovery. Lise, alone, remained clearly traditional. She could'nt make room for the child's intuition or for a discovery process in her teaching. Also she gave a rather behaviorist description of understanding.

The preceeding conclusions also apply to the subjects of the experimental group : simply replace Lise by Jacques... For Jacques did not change much. In fact, our constructivist beliefs led us to say that it will always be sheer utopianism to try to change someone's convictions againts its own will. The initial move or impetus has to be initiated by the person himself.

However, if the changes observed within the control group were small and isolated, the presence and coherence of those noted in the perceptions of five out of six of the experimental subjects reveal a real influence of the experimental course on these perceptions. For these individuals, the course was an opportunity to discover something new and/or to deepen and structure beliefs already present : concerning the function of intuition in the learning process, for instance, or the importance of analyzing children's errors. All this confirms and strengthens the indications obtained from the global analysis of answers to the correction test and to the questionnaire on school mathematics. However, the evolution observed here is not radical. And this is due to the fact that constructivist perceptions were more operative than expected at the beginning, but especially and above all, because real change in perceptions is more a continuous process than an event.

SOME FINAL REMARKS

Finally, what should be retained is that this research brings novelty in two ways.

- First, by its method. Because of this study, new tools to evaluate teachers' perceptions were built and tested. More importantly, the simultaneous use of these tools, each being very different from the other, allowed a search for converging indications which mutually validated themselves.

- Secondly, by its conclusions. In fact, the results of this study could lead to better teachers training programs where mathematics concepts would be considered simultaneously in a epistemological and psycho-pedagogical perspective. This particular kind of integration of different aspects of teaching and learning could help to solve the problems mentioned at the beginning of this paper because it would put the focus on the way concepts are known and understood.

Let us add that since this research was completed, a new model arose from the one just described (Herscovics and Bergeron, 1988). This new model is a two-tiered one, the first tier describing the understanding of preliminary physical concepts, and the second one describing the understanding of the mathematical concept itself.

Enlarged Constructivist Model of Understanding

Understanding of Preliminary Physical Concepts		
Intuitive Understanding	Procedural Understanding	Logico-physical Abstraction

Procedural Understanding	Logico-mathematical Abstraction	Formalization
Understanding of mathematical concepts		

This new model links up more explicitely than the preceeding one the chidren's mathematics to their physical experience and thus suggest more strongly to use the latter as a starting point in the construction of their mathematical knowledge. The new model also proved to be a better instrument for epistemological analysis. Using such analysis, teachers were able to develop tasks related to every aspects of understanding of a given concept, thus presenting a broader range of activities whose complementarity adds up to a much richer cognitive environment.

This leads us to believe that the conclusions of the research that were presented here are still valid when we use the approach that was developped with the new model instead of its precursor. Yes, it is possible to influence teachers' perceptions of mathematics and mathematics teaching and learning. Yet, the transformation that occured is not always as clearcut and drastic as one may wish. It is usually much more a small and slow evolutive process than a radical and sudden revolution. But if one thinks in terms of perennity, such an evolution might be more desirable than a quick alteration of the perception that will again be transformed by the first change that will occur in the direction of the wind.

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