Third Misconceptions Seminar Proceedings (1993)

Paper Title: Children's Constructions of Fractions and their Implications for Classroom Instruction
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Abstract:

Keywords: concept formation,,,misconceptions,,,,,
General School Subject: mathematics
Specific School Subject: arithmetic
Students: elementary school

Macintosh File Name: D'Ambrosio - Fractions
Release Date: 10-16-93 A, 11-5-1994 I

Publisher: Misconceptions Trust
Publisher Location: Ithaca, NY
Volume Name: The Proceedings of the Third International Seminar on Misconceptions and Educational Strategies in Science and Mathematics
Publication Year: 1993
Conference Date: August 1-4, 1993
Contact Information (correct as of 12-23-2010):
Web: www.mlrg.org
Email: info@mlrg.org

A Correct Reference Format: Author, Paper Title in The Proceedings of the Third International Seminar on Misconceptions and Educational Strategies in Science and Mathematics, Misconceptions Trust: Ithaca, NY (1993).

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Children's Constructions of Fractions and their Implications for Classroom Instruction[1]

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Researchers at The University of Georgia have been studying children's constructions of fraction schemes for three years as part of a project funded by the National Science Foundation. The NSF project, titled Children's Construction of the Rational Numbers of Arithmetic, involves a teaching experiment (see Steffe & Cobb, 1983 for a complete description) in which one teacher, generally a university faculty member or graduate student, works with two children in the environment of a computer microworld. Members of the NSF project staff have designed three computer microworlds for use in the project using Object Logo. One of the microworlds is called TOYS and involves the manipulation of various discrete objects. Another microworld, STICKS, allows the children to manipulate line segments, thus providing experiences with linear continuous units. The third microworld is called CANDYBAR and involves the manipulation of rectangular regions, or two-dimensional continuous units. (See Biddlecomb & Whitmire, 1992 for a more complete description of the microworlds.)

The same thirteen children are being followed through their third, fourth, and fifth grade years. We began to collaborate with their classroom teachers during the winter of their fourth grade year. Our primary purpose for initiating this collaboration was to explore ways of adapting NSF project findings to a whole class situation. There were many facets to the adaptation, including working with a class of 25 students of varying abilities instead of two students of matched ability, changing computer activities into activities without a computer, dealing with the limitations of physical materials and their accompanying classroom management problems, contending with district and teacher objectives for content coverage and assessment, and trying to be loyal to a constructivist philosophy of teaching within the constraints of a traditional classroom.

We began our experience by meeting with the three fourth grade teachers several weeks before they were scheduled to begin teaching the fractions chapters in their textbook. We discussed the teachers' objectives and concerns for the unit, and we shared with them some of our experiences and observations in the NSF project. We offered to assist the teachers in teaching the fractions unit in any way they wished, from observing, to providing manipulatives, to providing computer assistance, to teaching a few lessons. We were careful to emphasize that we did not want to take over their classrooms and teach the entire unit because we did not want to convey the idea that we knew how to teach the unit "correctly" and they did not. Each teacher had access to a Macintosh computer with the three microworlds on its hard drive, but the teachers had virtually no experience with these microworlds.

One teacher, Mrs. Watson, was particularly interested in having us in her classroom on a daily basis to observe her lessons, interact with the children, work with the children in small groups on the computer, co-plan lessons, and teach lessons. One or the other of the authors (and sometimes both) was present in Mrs. Watson's classroom every day for approximately three weeks.

In this paper, we will relate our experiences in Mrs. Watson's classroom. We will first identify specific findings that have emerged from the NSF project and describe how we attempted to address these findings in our work in the classroom. We will then discuss the children's constructions of fraction concepts that we anticipated seeing and the constructions we saw that we did not anticipate. We will also describe activities that were used in the classroom in an attempt to address students' limited constructions of fractions. These activities were often adaptations of activities used in the NSF project in the computer microworld environment. We will describe our attempts to design suitable off-computer activities and our struggles to align NSF project objectives and classroom teachers' objectives through computer and computerless activities.

NSF PROJECT FINDINGS

Iteration of the unit. After working with children for one and one-half years using the TOYS and STICKS microworlds, the NSF project staff had developed a set of working hypotheses about how children might be helped to make more powerful constructions of fraction concepts. Some of these hypotheses were in the form of suggestions for approaches to teaching fraction concepts, and others were in the form of statements about limiting constructions they had seen children develop. We tried to take these hypotheses into account when designing activities for classroom use, but as we will discuss later in the paper, the hypotheses did not always translate neatly into classroom activities.

The fundamental hypothesis of the NSF project staff was that children need to construct a definition of a unit fraction as an iterable unit which can be used to reproduce the whole or to produce any other fraction with that denominator. For example, a robust definition of 1/3 would be that 1/3 is that quantity which can be iterated three times to produce the unit. In addition, it is crucial to recognize that 1/3 can be iterated n times to produce n/3, even if $n > 3$. This contrasts sharply with the traditional textbook definition of 1/3 as one out of three equal pieces.

In the NSF project setting, children were introduced to fractions in the linear, continuous microworld of STICKS. As an introductory activity, children were given a stick and asked to mark where half of the stick would be and then determine whether or not they had found half of the stick. Children often used a strategy of comparing the two pieces of the stick to determine if they were equal. When the same problem was posed with marking 1/3 of the stick, the comparison strategy was no longer efficient. Children were then constrained to use an iterative strategy in which they copied the piece they had identified as 1/3 two times and checked to see if the stick joined with its copies was the same length as the original stick. Children later used this iterative strategy to make fractions such as 2/3 and 7/3.

Whole numbers to fractions. Closely related to the notion of a unit fraction as an iterable unit is the NSF project staff's hypothesis that children's constructions of fraction concepts should emerge from their constructions of whole numbers. Steffe (1992) has identified a variety of types of number sequences that children might construct. He states that children who have constructed a connected number sequence for whole numbers recognize one as an iterable unit which can be iterated to obtain any other whole number. Thus, the NSF project staff engaged children in fractional activities which had their basis in the children's whole number knowledge. For example, in the STICKS microworld children constructed a "staircase" of sticks ranging from a stick of length one (which was arbitrarily drawn) to a stick of length 12. Students were then asked three types of questions: Which stick would result if the 6-stick was repeated seven times? How many 8-sticks would be needed to make a 24-stick? Which stick could you repeat four times to make a 28-stick? Students could answer these questions using their knowledge of whole number addition, multiplication and division facts, and they checked their results by iterating the appropriate stick the correct number of times to produce the desired result. NSF project staff members then introduced fraction language into the activity by noting that since an 8-stick could be repeated three times to make a 24-stick, an 8-stick is 1/3 of a 24-stick.

Traditional algorithms for finding equivalent fractions, simplifying fractions, comparing and ordering fractions, and performing operations with fractions rely heavily on children's facility with basic facts involving whole numbers. However, in our experience we were more interested in helping children use their whole number knowledge to develop meaning for fractional concepts than for performing algorithms.

Linear models. Traditional textbooks and classroom practices make minimal use of a linear model for fractions. Their use is generally limited to using a number line to compare fractions. However, in the NSF project, children spent the first year working with the discrete TOYS microworld and the second year working with the linear STICKS microworld. In their third year, the children will begin to work with the area microworld, CANDYBAR. The reasoning behind this sequencing of experiences was that linear models help children construct rich definitions of fractions without becoming dependent on the "number of parts shaded out of total number of parts" view of fractions typically afforded by the area model. In our experience, we engaged the children in activities involving discrete, linear and area models for fractions, and we placed particular emphasis on activities involving folding strips of paper and making number lines to familiarize them with a linear model for fractions.

Estimation and power of visual representations. As children in the NSF project worked with the computer microworlds, it became apparent that they were empowered by the dynamic visual representations afforded by the microworlds. Having the capacity to copy, join, cut, erase, and mark objects instantaneously proved to be a powerful and liberating experience for the children. They were able to engage in estimating, predicting, testing, and revising without any threat of the permanence of their actions. Further, since the microworlds are devoid of any numerical data unless requested by the user, the children were able to rely heavily on visual estimates, which seemed to strengthen their concept of fractional quantities.

In designing classroom activities, we were limited by the availability of computer equipment, so we tried to simulate computer activities with concrete manipulatives such as paper strips, pattern blocks, and M & M's. We quickly discovered that these physical materials have limitations which make it impossible to replicate computer experiences. Thus, we began to work with the children in groups of five to eight in the hallway with the computer to afford them some of the same experiences that children were having in the NSF project.

SOURCES OF STUDENTS' LIMITING CONSTRUCTIONS

Based on the findings of the NSF project, the findings of other research projects, and our own classroom experiences, we anticipated that fourth graders would have constructions about fractions that would limit their further understandings. We have intentionally avoided labeling the children's constructions as misconceptions because we believe it is important to view children's constructions as objects for study which can provide insight into and new understanding of their thinking. We felt that it was important to take children's constructions into consideration when planning classroom activities, so we endeavored to design activities, both with and without the computer, to explore and challenge the children's constructions. In doing so, we uncovered children's constructions that we did not anticipate.

We will begin our discussion by summarizing some of the children's constructions about fractions and speculating about how they may have emerged. Then we will describe classroom activities which we designed to extend and modify the children's constructions. We will detail how these activities were actually implemented in the classroom and what new constructions emerged.

Some of the constructions about fractions that we anticipated were the following: numerators and denominators are viewed independently in making meanings for fractions; counting is a fool-proof method for determining fractional amounts; "equal" parts are parts that look identical to each other; there is no connection between division and fractional representations of quantities. We have sorted these constructions by what we speculate to be their potential sources. These sources include the language of instruction; the visual representations used, including the types of examples and non-examples used; and the textbook sequencing of topics. Taken together, these components roughly describe the children's prior experiences with fractions.

Language of Instruction. As early as first grade, children are presented with examples that lead them to construct a definition of a fraction as number composed of a numerator and a denominator which represent different quantities that can be determined by counting. Children often see little relationship between the numerator and denominator in a fraction representation of a quantity. The definition "number of parts shaded/number of equal parts in which the shape was subdivided" only begins to have meaning to children as they look at examples and work on several exercises that reinforce the "definition" used in the book. Most of the examples used in early work with fractions suggest to children that a fraction is merely the result of two counting experiences reported in a special format.

Another aspect of the language of instruction that seems to influence children's constructions is related to their interpretation of the term "fair." The notion of fair shares, often used to describe and discuss fractional representations, may have different meanings for the child and the teacher. Although used by the teacher to emphasize equality in partitioning, children's real world experiences seem to interfere with this interpretation. NSF project staff members routinely used the phrase "mark this stick in three pieces so that the shares are fair" with children to indicate that the stick was to be divided into three congruent pieces. The word "fair" was generally substituted for "equal" or "congruent" throughout NSF project teaching sessions, but it became apparent that the children's meaning of "fair" was not the same as the teacher's. Presumably based on their real world experiences, children seemed to define "fair" as "everybody gets a piece of approximately the same size." This was particularly true when the children were partitioning representations of food items. It seemed natural to them to divide the items into "fair enough" shares. The word "fair" did not carry the connotation of precision or equality that it carried for the adults. The notion of sharing apparently did not correspond to children's prior work with partitive division either.

Also related to the language of instruction are the difficulties students seem to have with the notion of what constitutes the whole in different problems or experiences. During instruction, teachers often take for granted that the whole in consideration is obvious. Little time is spent discussing how fractional representations for the same figures might differ if the whole was defined in different ways. For example, when asked to shade one fifth of the objects below (Figure 1), children may wonder what constitutes the whole.

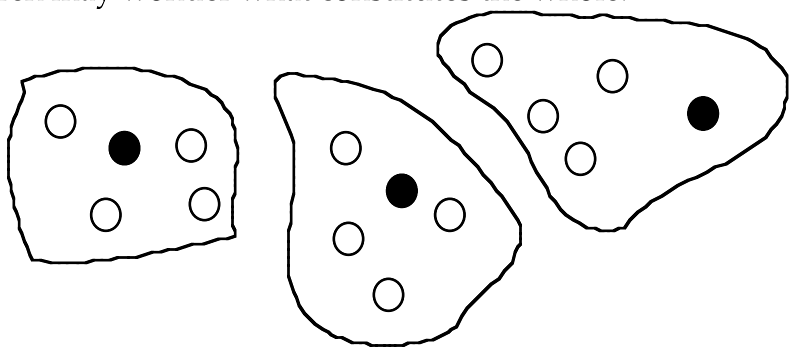

Figure 1

It is not unusual for children to partition the objects into sets of five and shade one in each set. Although three objects will be shaded in the end, this may be an indication that the children are struggling to accept the entire set as a whole.

Visual representations. Much of children's early work with fractions leads them to a stongly held construction that "equal" means "look alike." There are few examples used in children's early work with fractions that would falsify this belief. In fact, most non-examples of the belief are used as non-examples of fractional representations, which serves to reinforce their belief. Rarely will a child be asked to reason whether the figure below (Figure 2) would represent a whole sectioned into fourths.

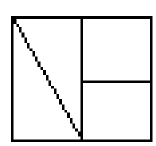

Figure 2

Even discrete models are usually drawn with multiple pictures of identical objects. The discussion about whether the size of these objects is important is never raised. In fact, in one textbook series analyzed, the question is trivialized by a definition of fractional representations that states that the child should be concerned with the size of the parts only when dealing with the area model.

In examining textbooks it is interesting to note that half of a figure is almost always represented by shapes that are cut into two parts or a number of parts that represents a multiple of two, and half of those parts are shaded. Rarely will a child be asked to identify what fractional part of the figure below (Figure 3) is shaded. Note that in this figure there are three component parts, and yet the shaded region is in fact one-half of the total figure. The mere existence of three parts would be a distracter for many children and, based on their constructions, most children would conclude that the shaded part could not be represented by a fraction, especially not a fraction where the denominator is two.

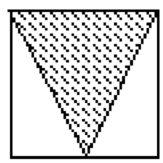

Figure 3

Another common construction by the children, that we speculate originates from the types of pictorial examples used in the textbooks and lessons, is in their understanding that sections shaded should be adjacent to each other. Although many children would recognize two-sixths as shaded in Figure 4, it is not unusual for children not to recognize two-sixths as shaded in Figure 5.

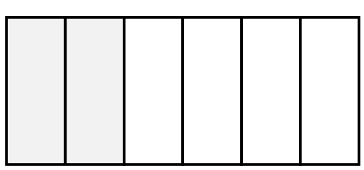

Figure 4

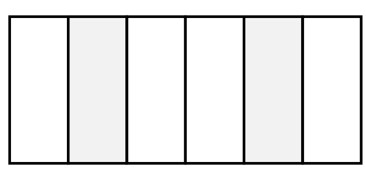

Figure 5

Textbook sequencing. In order to understand the nuances involved in pictorial representations of fractional quantities, children need a thorough understanding of area and equivalence of area. Connections to other areas of the mathematics curriculum are also crucial to developing a rich understanding of fractions. For example, linking work with fractions to linear measurement can aid children in learning concepts, such as the iterative unit, and skills, such as the juxtaposition of units to measure a segment. Unfortunately, the mathematics curriculum is often presented in a compartmentalized fashion so that children do not have the opportunity to construct connections between various topics, such as fractions and geometry and measurement. Typically, chapters on geometry and measurement are placed later in textbooks than the fractions chapters. Even when these chapters appear earlier in the book, they tend to be skipped and "saved" for after achievement testing.

Another troublesome aspect of textbook treatment of fractions is the omission of experiences with the number line model. In one textbook series the number line is mentioned for the first time in the fourth grade book, and then only briefly in discussing comparison of fractions. Interestingly, the NSF project has found the linear model to be quite effective in children's construction of fractional concepts, especially in the use of the unit fraction as an iterative unit that can be used repeatedly in order to build the whole. The linear model tends to de-emphasize the need for understanding relationships of area as children are striving to make sense of fractional constructs.

ACTIVITIES

Classroom activities were designed with the aforementioned constructions and sources of limiting constructions in mind. The first set of activities involved experiences with pattern blocks. The intent of these activities was to provide children with experiences that would help them construct meaning for the concept of "equal parts" embedded in the typical definition of fractions used in textbooks. Since the children had not had recent experiences with notions of equivalence of area, we did not want to assume that they could draw on that construction for understanding fractional representations of shaded regions. We hoped that through our interactions with the children during the activities we would come to better understand their constructions of equivalence of area.

In their initial work with pattern blocks, the children were asked to fill certain regions with the blocks. Each region was then considered the whole, and the blocks were used to describe different fractional parts of that whole. The emphasis of the discussion on the first day of instruction was on the number of different ways of filling the outlines of the shapes. Children counted the shapes needed to fill the outline and reported with whole number descriptions. For example, to fill a region in the shape of a hexagon the children reported the following:

C1: I used two reds.
C2: One red, three greens.
C3: Six greens.
C4: One red, one blue, one green.

The chance to introduce alternative language involving fractional terminology for these experiences was missed. However, in analyzing this episode we came to realize how the use of the fractional language for these examples conflicted with the teacher's view of how fractional language should emerge. Typically, she would have used examples in which the numerator would clearly indicate "the number of parts shaded" and the denominator would indicate "the total number of parts in which the shape was divided." This definition was only directly applicable when pattern blocks of one color were used.

Just a few days later the following question was asked of the children:

T: Who can explain the meaning of 4/5?
C1: Its like a pie divided in five pieces.
T: How many pieces would you have?
C1: 4
T: Good, four-fifths means you would have four pieces out of five.

This "review" reflects the limited view of the use of fractional language as a means of representation that was portrayed to the children. This is especially worrisome considering that it followed work with pattern blocks in which 1/2 had been represented in numerous ways, many of which included more than 2 pieces. The opportunity to probe for children's constructions and try to understand their perceptions of work with those materials was missed. What might have served as a powerful perturbation for a limiting construction was overlooked.

A situation emerged that indicated a conflict between the teacher's predetermined conceptions of what she was looking for and the children's solutions to the situations posed. Children were asked to share their solutions for covering the yellow hexagon from the pattern block set. The following solutions (Figure 6 and Figure 7) for using two blue pieces (parallelograms) and two green pieces (triangles) were placed on the overhead projector:

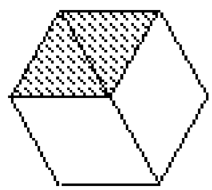

Figure 6

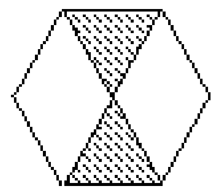

Figure 7

The children were told that these were the same, and Figure 7 was removed from the overhead projector. Here again the opportunity to challenge a limiting construction was missed. The knowledge that children often have difficulty naming fractional parts that are not adjacent would justify a discussion with the children of why the two green pieces in Figure 6 and in Figure 7, in both cases, represented two-sixths of the covered shape. This idea was taken for granted and may have been an important point to pursue.

The second set of activities involved work with paper folding. The use of paper folding resulted in several interesting observations. Although paper folding was initially suggested as a means of simulating the STICKS microworld, the limitations of the concrete model soon became evident. First, the strips of paper were not interpreted by the children as linear models. They had two dimensions, and the children suggested that the strips could have been folded in another direction.

As the paper folding activities proceeded, it soon became apparent that the children were often using additive reasoning skills to predict the number of pieces they would find upon opening a repeatedly folded strip. For example, after folding a strip in half and in half again to produce fourths, the children were asked to predict what would result after another fold. Many of the children predicted that the paper would be divided into sixths rather than eighths. This construction would later greatly limit their understanding of equivalent fractions.

The inaccuracies of the folding process severely limited the possibility of using the folded piece as an iterative unit to generate the whole. For example, the children folded the paper in half, half again and repeated the process until they had sixteenths. The question was then, what would be more effective, unfolding the strip and counting the pieces or using the one sixteenth as an iterative unit to replicate the whole? In an attempt to simulate the STICKS microworld we decided to try the latter experience. Whereas in the STICKS microworld the child might estimate a sixteenth and make fifteen copies of the estimate resulting in 16 identical pieces, with paper folding this was not possible. When iterating the sixteenth, the resulting unit was often larger than the original strip. Interestingly, this did not generate any perturbation for the children, which led us to realize that many of them had apparently not yet constructed the iterative fraction scheme. Unfortunately, the inaccuracy of the materials did not allow the children to engage in activities that would help them construct iterative fraction schemes and unit partitioning schemes (Steffe & Spangler, 1993; Olive, 1993).

The paper folding activity and the forms used to record children's findings in their explorations revealed yet another unanticipated limiting construction. When unfolding a paper strip and identifying the result, the children were apparently labeling points, as they would on a number line, and losing sight of the notion that the folded strip represented a fraction of the original strip. The work with paper folding and the labeling of the parts was disassociated from the experiences with fractions to that point. For the children working with the microworld in the NSF project it became clear that any combination of x pieces of a certain size (say $1/4$) would produce $x/4$. However, with the paper folding that construction was not made, and, in fact, the notion of parts having to be adjacent to determine a fraction was reinforced.

Throughout the fractions unit, time was provided for children to work in groups of four to eight on the computer with the microworlds with one of the researchers present. All of the children in the class except one were unfamiliar with the microworlds. One child in the class was one of the subjects of the NSF project study, so he had worked with the TOYS and STICKS microworlds for two years. Most of our work was with the CANDYBAR microworld, which he had not used. The nature of the children's experiences with the microworld was quite different from the experiences provided in the NSF project due to the children's lack of familiarity with the microworlds, the size of the groups we worked with, and the time constraints under which we were working.

The children's tendency to use additive reasoning in multiplicative situations in paper folding activities was also manifested in computer activities in the CANDYBAR microworld. When asked how to make a candybar with 12 pieces, they replied that making six vertical pieces and six horizontal pieces would yield a candybar with 12 pieces. When they actually performed this operation in the microworld and produced a candybar with 36 pieces, they were puzzled. We repeated this activity many times and tried to find all possible ways of making a candybar with 12 pieces.

An activity involving equivalent fractions demonstrated the power of the visual imagery afforded by the microworlds. The children had already had classroom experiences with paper folding (both strips and sheets), pattern blocks, and the pencil and paper algorithm for finding equivalent fractions when this activity took place on the computer. The children were asked make a candybar with 12 pieces and to fill in some of the pieces. They were then asked to give all names for the fractional amount they had shaded. At one point, the computer screen showed a candybar with six adjacent pieces shaded as shown in Figure 8. The children immediately said that 6/12 of the figure was shaded. Seconds later someone noted that 1/2 of the figure was shaded. Then someone said that if "you pretend that these lines (the horizontal lines) aren't there, 2/4 of it is shaded" (Figure 9). To demonstrate what this child was thinking, we constructed a new candybar of the same size with four vertical pieces in it and shaded two of them (Figure 9). The children were then presented with several more problems of this type, and they were able to imagine removing both horizontal and vertical markings to find other names for the fractional amount that was shaded.

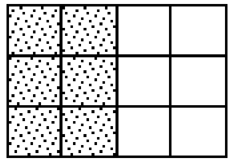

Figure 8

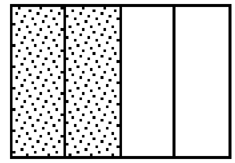

Figure 9

Later in the same lesson, the children were shown a candybar with eight randomly selected pieces shaded (see Figure 10). When asked for a name for the fractional amount shaded, the children quickly counted the number of shaded pieces, and said 8/12. When asked for other names for the fractional amount shaded, the children were silent for a while. Then one boy said, "Oh, I know! It's 2/3!" He explained that he imagined moving the shaded pieces into a configuration where they completely filled a row or a column, thus reducing the problem to an earlier one which we had already solved. Figures 11, 12, and 13 show the child's thinking as he described it.

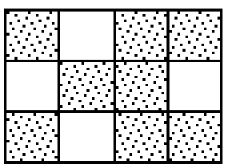

Figure 10

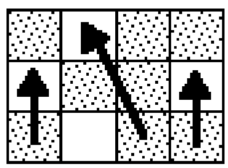

Figure 11

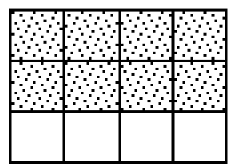

Figure 12

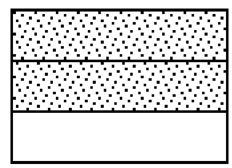

Figure 13

Throughout the activities posed for the children we observed then relying heavily on their counting schemes. To challenge the children's definition of a fraction as a number which is the result of two counting acts, we designed a computer activity in which they could not count to solve the problem posed. The children were presented with a candybar and asked to use the shade function of the microworld to shade some fraction of the bar. The shade function of the microworld works like a window shade; the mouse pointer is placed at the left or bottom of the candybar, and as the mouse is dragged across the screen, a shading appears. The shade can be pulled completely across the bar and can be retracted without leaving any evidence of where it has been. Once a shade is put in place by releasing the mouse button, it can be moved again by simply holding down the mouse button and dragging the end of the shade. This function allows children to estimate and revise their estimates without penalty.

This activity began with the children shading unit fraction amounts of a given candybar. The children were able to check their estimates by comparing them with bars already shaded (i.e. 1/4 should have less shaded than 1/3) and by having the computer superimpose a partition of the bar into the desired number of pieces on top of their shaded bar. The children were allowed to revise their estimates as many times as they desired.

Throughout this activity, the children demonstrated an understanding of the concept of a unit fraction as an iterable unit. When asked to shade 1/4 of a candybar, many of the children would pull the shade to a spot that they thought might represent 1/4, measure this amount with a thumb and index finder, and iterate this measurement three more times to see if it filled the whole bar. If not, they would adjust the shade slightly and repeat the process until they were satisfied with their estimates. When asked to shade 3/5 of a candybar, one child pulled the shade to where she thought 1/5 would be, measured this amount with her thumb and index finder, iterated this measurement two times, and pulled the shade over to the spot marked by her finger.

In the final set of classroom activities we used fractional language to describe sets of discrete objects. The first experiences were geared toward using fractions to describe the nature of the class membership. The number of males and females was compared and represented in fractional language. This was done exclusively as a counting experience and reinforced the use of counting schemes. That activity was followed by describing the contents of a bag of pattern blocks. Again, this was mostly a counting activity where the children were reporting that x blocks were yellow, y blocks were red, etc. The teacher would then reinterpret those numbers in fractional language.

The final activity involved the use of M & Ms. First, children were asked to determine the fractions representing the ratio of M & Ms of a particular color to the total number of candies in their bags. After reporting the ratios for each color, the children were asked to count out 24 candies. They were asked to make three equal groups.

T: What is 1/3 of 24?

Several children: 8/24

This solution revealed a dimension of their understanding that we might have otherwise overlooked. The logic of this response indicates a natural transfer from one activity to the next. While the group they were describing could be called one-third, it might also be called eight twenty-fourths. The connection between finding one-third of 24 and dividing 24 by three was never made. Instead the children related their actions to that of describing one-third using a different fraction. They were using their counting schemes to reinterpret the name given to the group. The whole number answer the teacher was insisting upon made no sense to the children whatsoever.

This is perhaps the example which most clearly illustrates the purpose of a teaching experiment in a constructivist framework. The children's responses generated a need for the teachers to interpret the underlying mathematics that the children were using. This process of searching for meaning in the children's work led us to reconstruct our own understanding of the underlying mathematical concepts involved in the activity. In this particular case we were taken aback when we recognized the children's struggle to understand the whole in consideration. The tremendous shift in interpretation of what constitutes the whole was leading to a gap between the teachers' interpretations of the problem and the children's interpretation. Interestingly, we often take it for granted that the assumption of what constitutes the whole is understood. Up to this point the children had worked with describing the set by using fractional names for the subsets. Suddenly, they were asked to give a whole number result for what seemed like the very same problem - describing a subset of a given set. There seems to be a paradox - if the set of objects is considered the whole, then how can each individual object be named as a whole?

FINAL DISCUSSION

Two themes have emerged as we have reflected on our experiences in the classroom. The first deals with children's constructions of fraction schemes, and the second deals with what we have learned about the dynamics of the teaching-learning process.

From our constructions of children's fraction schemes, we have been able to conclude at least four things. First, developing a conceptual understanding of fractions takes time. This is certainly not a startling new finding, but it has strong implications for teaching fractions. In a typical classroom, one week is devoted to the "introductory" chapter on fractions. In our experience nearly three weeks of 45 minute lessons were spent on developing fraction concepts, and yet we only scratched the surface of the children's potential constructions. We also likely didn't even begin to uncover all of their limiting constructions. In contrast, the children in the NSF project spent an entire year working with the STICKS microworld and introductory fraction concepts.

Uncovering children's limiting constructions was another learning experience for us as teachers. Several of the constructions discussed earlier in this paper were uncovered quite by accident. A chance comment by a particular student or an opportune observation of a particular student's work often suggested a line of questioning to pursue that led to students verbalizing their constructions of the material being discussed. Allowing children to verbalize their thinking, however painful that may be for them, the teacher, and the rest of the class, provides irreplaceable opportunities to gain insight into students' thought processes. We were amazed by the number of limiting constructions the children held and how often they were uncovered purely by chance. Our experience suggests that children potentially hold a large number of limiting constructions about fractions and teachers need to make a conscious effort to allow children to verbalize their thinking rather than "filling in the blanks" for them and assuming that we know what the child "really means."

The M & M example discussed earlier showed us that children try very hard to make sense of what goes on during a mathematics lesson, and they impose structure on situations that seem to them to be unstructured. When no structure is obvious to the children, they tend to overgeneralize or impose the most recent structure used on the unfamiliar situation. Such was the case with the M & M activity where the children assumed based on their recent experience, that the answer to the question, "What is 1/3 of 24?" must be a fraction. This experience suggests that children need rich and varied environments in which to construct fraction concepts. This richness and variety can come in the form of concrete materials, computer access, interesting and challenging problems, group work, whole class discussions, and many other forms.

In comparing and contrasting the experiences of children working with the microworlds and the experiences of children working with concrete objects such as paper strips and pattern blocks, we have concluded that both types of experiences have their limitations. Since the microworlds do not pose problems for children, it is crucial (at least in the initial stages) for a teacher to be present to pose problems and probe children's responses. The ideal setting for working in a microworld would be two children and one teacher per computer. However, since present school circumstances do not allow for such luxury, children's ownership of the learning experience is severely limited when they must share computer time with six or eight other students.

Some of the limitations of the concrete materials have been alluded to in previous sections of the paper. Many manipulations with concrete materials are somewhat permanent and cannot be easily changed. Other materials invite errors that may lead to limiting constructions because of their lack of precision.

In retrospect, one of the most striking differences between teaching in the NSF project activities and teaching in the classroom setting was in the role of the teacher. Our intention in the NSF project and in the activities proposed for use in the classroom was to create a teaching-learning situation in which we would seek to understand the students' meaning making from the experiences we invited them to engage in and find ways of modifying these meanings when necessary.

Another characteristic of such an environment is the nature of the experiences created for the students' involvement. The deemphasis on direct instruction and the greater emphasis on student activity characterized another of our goals. We agree with Papert (1990) when he claims that: "Instruction is not bad but overrated as the locus for significant change in education. Better learning will not come from finding better ways for the teacher to instruct but from giving the learner better opportunities to construct."

The teachers in the NSF project ask questions that elicit children's verbalization of constructions or actions upon objects that reflect their constructions. The teacher in the classroom elicited responses of a very different nature. Questions were intended to be evaluative in that they indicated whether she could move on or whether further instruction or practice was needed. There was little attempt to probe and understand children's meanings. Her purpose for working with us was, in fact, to seek better ways of instructing her students.

In conclusion, we feel that substantial work is needed with teachers to help them reconceptualize their roles in the teaching and learning process. This may enable them to teach in a way that leads to greater understanding of how children construct knowledge, so that they can use that understanding to provide experiences that help children interact with their world and interpret it using their own mathematics and the power it carries.

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[1] The experience reported in this paper was supported in part by the NSF Project No. RED-8954678, Children's Construction of the Rational Numbers of Arithmetic. All opinions expressed are solely those of the authors.