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Author: Baturo, Annette R.

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A Cognitive Analysis of Misconceptions in Year 12 Students' Understanding of Elementary Probabilistic Notions

Annette R. Baturo
Centre for Mathematics and Science Education
School of Mathematics, Science and Technology Education
Faculty of Education
Queensland University of Technology
Brisbane, Australia

This paper describes a study conducted to explore Year 12 (16-17 year olds) students' cognitive functioning in the domain of probability in an endeavour to discover what it means to know/understand the elementary notions of probability. Leinhardt's (1988) theory of understanding as connections between the four knowledge types (*intuitive, concrete, computational* and *principled conceptual*) served as the model for examining the students' understanding of elementary probability. The research design incorporated two pilot studies and a main study and, altogether, 31 students participated. Each student was clinically interviewed whilst working on a set of elementary probability tasks which were developed for the study. The protocols revealed that the students had used a variety of cognitive schema for example, fraction (*part/whole)*, ratio (*part/part)*, and comparison (*whole/whole*) but, in general, those who performed best used the fraction schema predominantly. Several misconceptions were disclosed. For example, P = 1 was connected with either one trial in an experiment or with one item in a sample space; P = 2 was acceptable; *possible* was synonymous with *certain*; ratios were confused with fractions.

Probability is the branch of mathematics that describes randomness, an attribute that should not be associated with "haphazardness" but rather with a kind of order that is different from the deterministic one normally attributed to mathematics (Steen, 1990, p. 98). An understanding of probability is essential in a culture whose technological advances in all fields of the physical and social sciences have given rise to a probabilistic, rather than a deterministic, view of the nature of mathematics (Ashmore, Dodge & Kasch, 1962, p. 530). This shift in ideology from determinism to probabilism has been echoed in recent national and international educational documents reporting on mathematics curricula suitable for the 21st century (Australian Educational Council, 1990; Carl, 1989; National Research Council, 1989; Steen, 1990; Thompson & Rathmell, 1988).

There has also been a transition in social paradigm from Industrial Age to Information Age (Romberg, 1987) and this has resulted in a wider definition of numeracy to suit today's society. Rather than simply meaning computational ability, Cockcroft (1982, p. 11) has stated that, to be numerate, people must have an "at-homeness" with numbers, an ability to make use of mathematical skills required for coping in their everyday lives and an appreciation and understanding of information that is presented in mathematical terms (e.g., charts, graphs, tables). The Australian Educational Council (1990, p. 82) believes that "sound concepts in the areas of chance, data handling and statistical inference are critical for the levels of numeracy appropriate for informed participation in society today".

Apart from its philosophical, cultural and social significance, probability, with its experimental nature and its emphasis on inquiry, is pedagogically suited to the contemporary educator's belief that children are active constructors of their own knowledge (von Glasersfeld, 1987; Narode, 1987) and that knowing is more important than knowing about (Romberg, 1983). Moreover, probability activities based on games of chance foster positive affective traits such as persistence, resourcefulness and a spirit of inquiry. Therefore, it is for all of these reasons that modern educational councils are strongly recommending the inclusion of probability in any mathematics curriculum designed for the 21st century (Australian Educational Council, 1990; Carl, 1989; National Research Council, 1989; Steen, 1990; Thompson & Rathmell, 1988). Moreover, these same councils recommend that the study begin in the very early primary years, a circumstance that may help to overcome the erroneous beliefs that some students develop concerning chance events and mathematics *per se*.

For example, from the cradle on, children are virtually saturated with the language of probability (e.g., *the chances are . . ., it's quite/highly likely that . . ., there's a 50-50 chance of)* . but if they are taught neither the concepts from which the language is derived nor how to deal realistically with uncertainties, then they "may respond to these situations with preconceived notions, emotive judgments and even a lack of awareness that chance effects are operating" (Jones, 1979, p. 37). This hypothesis is supported by other studies (Pedler & Robinson, 1977; Williams & Shuard, 1982) in which the findings indicated that children with no probability experience often believe that a guiding force (such as the way a die is rolled) controls the outcome of random events.

Furthermore, mathematics is concerned with uncertainty as well as certainty yet only the "certainty" type of mathematics is dealt with in the primary years. Jones (1974) postulates that children who are not exposed to uncertainty early in their mathematics education may be reluctant to accept that mathematics is not always correct, thus gaining a false impression that, in mathematics, there can be one correct answer.

In their review of the literature concerning children's intuitions and conceptions of probabilistic notions, Hawkins and Kapadia (1984) stated that, because of differences in the terminology and methodologies employed, research has generated debate rather than provide answers to the questions posed. They have identified and defined the following types of probability (p. 349).

- A *priori* (or theoretical) probability (obtained by making an assumption of equal likelihood in the same sample space.
- *Frequentist* probability (calculated from observed relative frequencies of different outcomes in repeated trials.
- *Subjective/Intuitive* probabilities (expressions of personal belief or perception.
- *Formal* probability (that which is calculated precisely using the mathematical laws of probability.

Hawkins and Kapadia believed that these different types of probability had important philosophical and psychological implications for research and, therefore, for the classroom. For example, they believed that subjective probability (which may rely merely on comparisons of perceived likelihoods) was an area which was often neglected in the classroom although it may be a fundamental precursor to formal probability (which requires some acquaintance with fractions).

Other earlier studies support this notion of a natural sequence in acquiring probability knowledge. For example, both Davies (1965) and

Piaget and Inhelder (1975) saw *chance* and *probability* as being two distinct notions. Davies referred to the estimation of chances as the *intuition of probability* and to the calculation of chances as the *concept of probability*.

Piaget and Inhelder (1975) conducted a series of experiments focusing on *a priori* probability with children of varying ages who conformed to their categories of *preoperational* (approximately 4- 7 years), *operational* (approximately 7- 11 years) and *formal operational* (approximately 12 years on). From the results of these experiments, they hypothesised that, while the notion of chance appeared at the beginning of the second stage of cognitive development (i.e., from 7 years to 11 years approximately), true intuitions of probability emerged only when a system of distributions was established and this was not until the child had reached the formal logical stage of development (i.e., after 11 years approximately). By inference, then, Piaget and Inhelder's view was that probability concepts should not be introduced to very young children because they did not have the cognitive structures necessary for processing this information.

This inference would appear to be supported by the results of the National Assessment of Educational Progress (NAEP) in which three key probability concepts were assessed (the probability of an event, the probability of independent events, and the probability of compound events (Carpenter, Corbitt, Kepner, Jr., Lindquist & Reys, 1981). To assess understanding of the probability of an event, the students were given a sample space (numbers on a set of ping pong balls (2, 3, 4, 4, 5, 6, 8, 8, 9, 10) and the desired event (drawing, blindfolded, a ping pong ball with a 4). Fifty per cent of 17-year-olds answered $^1/_5$ or its equivalent compared with only 32% of 13-year-olds. The most common "incorrect" response made by the 13-year-olds was "don't know" (16%, compared with 7% of 17-year-olds) followed by $^1/_{10}$, then $^2/_8$ (which was the next most common incorrect response made by the 17-year-olds). It is easy to see the influence of set theory in the response, $^1/_{10}$, as it appears obvious that the children counted the "identical elements" as one unit. The other response, $^2/_8$, indicates a ratio method of processing (2 outcomes are favourable, 8 aren't. Carpenter et al. (1981) believed that these results suggested that many students had some general concept of probability but did not know conventional means of reporting probabilities. Furthermore, the assessment revealed that the students found it more difficult to calculate the probability of a non-occurring event (an impossible event) than they did a possible event.

However, Fischbein (1975) and Jones (1979) claim that children as young as 5 years of age have the cognitive structures required for processing elementary probability concepts such as sample space and the probability of an event's occurring. In Fischbein's study, preschool, Grade 3 and Grade 6 children were tested on probability tasks His main purposes were to explore systematically the evolution of probability notions across Piaget's three major stages of development (preoperational, concrete operational and formal operational), the effects of prior systematic instruction, sex differences and the influence of the total number of outcomes on the correctness of responses.

In preliminary experiments, it was noticed that the subjects across all year levels tended to **estimate** the probability of an event by using *ratios* (number of favourable outcomes as opposed to the number of unfavourable outcomes) rather than *fractions* (number of favourable outcomes out of the total number of outcomes in the entire sample space). Fischbein (1975, p. 176) used the ratio and fraction procedures to differentiate between *chance* and *probability*, intimating perhaps that chance is a subjective informal estimate of an event whereas probability is the objective formal measurement of an event. The results of his study disclosed that preschoolers, both before and after instruction, seemed to base their responses on "spontaneous perceptual sampling" (p. 186). It was also found that the Grade 3 children's strategies, before instruction, did not differ from the preschoolers' strategies but, after instruction, they were able to compare the ratios correctly and thus their responses became comparable with those of the older group. Piaget and Inhelder (1975) had concluded that this concept of proportion was accessible only at the formal operational stage. Fischbein suggested that effective instruction could set up "structures corresponding to formal operations already at the concrete operational stage with much greater ease and more stability than would be the case for the transition from the preoperational to the operational stage" (p. 187).

Lovett and Singer's (1991) findings about young children's preferred use of perceptual strategies echoed Fischbein's (1975) findings. In their study designed to determine how young children generate probability estimates, Lovett and Singer concluded that many children have a non-quantitative understanding of probability, preferring to use perceptual strategies when both perceptual and quantitative strategies are supported. As a consequence of this finding, Lovett and Singer suggested that the relationship between perceptual and quantitative strategies needed to be explored to determine whether perceptual strategies eventually lead to quantitative strategies or whether the two strategies developed separately without ever interacting.

Jones's (1974) study involved individually interviewing Years 1, 2 and 3 students as they worked on a set of tasks designed to assess their understanding of the following elementary probability notions: (1) the comparison of events in one sample space; (2) the comparison of events across three sample spaces with the same number of outcomes in each (i.e., a/x, b/x, c/x), and (3) the comparison of events across three sample spaces with a different number of outcomes in each but with the same number of favourable outcomes (i.e., a/x, a/y, a/z).

Jones found that the Year 1 children performed poorly on all of the tasks but the Years 2 and 3 children performed badly on the last task only (the third elementary notion) which involves an ability to compare fractions. From this, he concluded that severe limits existed with regard to young children's probabilistic thinking. Jones also discovered that the type of material used to portray a probability task had an effect on promoting the probabilistic thinking of young children. He classified material according to whether they incorporated *set* (i.e., discrete) embodiments or *measurement* (i.e., continuous) embodiments, whether they allowed *gross* or *unit* comparisons to be made and whether outcomes were presented in a *contiguous* or *noncontiguous* form. (See Figure 1.)

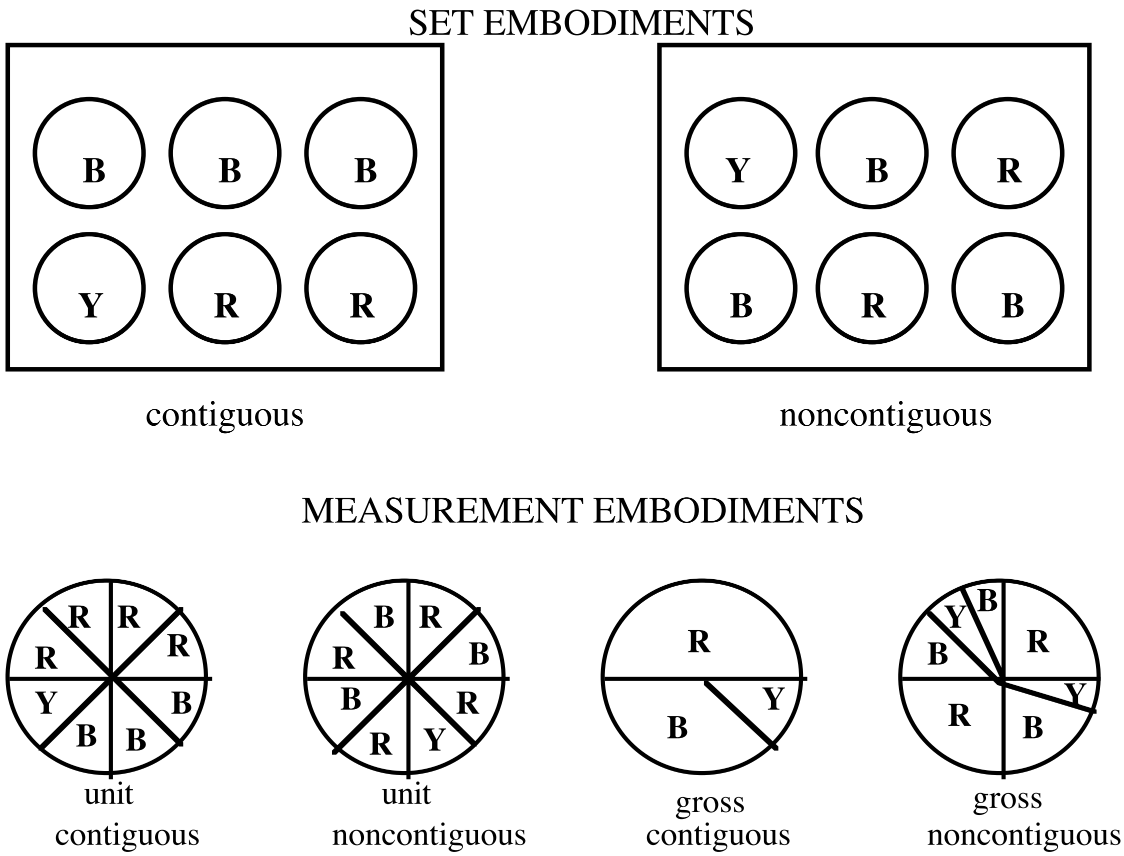

Figure 1. Jones' (1974) classification of concrete representations used in developing probabilistic thinking. (Note that set embodiments can only be unit representation.)

Jones discovered that set embodiments (e.g., marbles) were more effective than measurement embodiments (e.g., spinners) in promoting probabilistic thinking, that children found the unit model easier to interpret than the gross model and contiguous representation easier to interpret than noncontiguous representation.

As revealed by the literature, the ability to calculate the probability of an event's occurring can be limited by an inadequate understanding of, and facility with, fractions. As well, the literature revealed that many students use a spontaneous perceptual strategy which appears to be related to ratio structures. Therefore, it seems worthwhile to describe these probability-related domains from a pedagogical perspective.

A *fraction* is a generic term used to denote a numerical amount that is a *part of a whole* (Kieren, 1983; Nik Pa, 1989; Payne, Towsley & Huinker, 1990, p. 178; Smith, Booker, Cooper, & Irons, 1987, p. 17). The *whole* can be any continuous quantity (e.g., a region/area, a line or a volume) or discrete quantity (e.g., a set of objects).

Most modern mathematics syllabi (Department of Education, Queensland, 1987a; Australian Education Council, 1990) advocate the use of the area model in developing the understanding of a fraction because it has long been known that children experience conceptual and perceptual difficulties in interpreting the other models (Payne, 1976). For example, with the set model, the whole and the equal parts may not always be evident. With the linear model, children tend to see the marks as discrete points on a line instead of as parts of a whole unit/segment whilst the volume model does not enable the equal partitions to be shown. Although the set, linear and volume models are not used to develop the understanding of fractions, they should not be avoided as full understanding of any notion requires an ability to abstract the salient features from a variety of materials (Dienes, 1969).

A fraction can be recorded in several different ways (informally (e.g., 3 tenths), as a common fraction (e.g., $^{3}/_{10}$), as a decimal fraction (e.g., 0.3) and as a per cent (e.g., 30%). However, it should be noted that the concept of a fraction as a part of a whole remains unchanged whatever the method of recording.

In the current age of metrication, the decimal-fraction recording is essential and, in view of this, current mathematics syllabi recommend that decimal-fraction recording be taught before common-fraction recording whilst the recording of per cents should be delayed until children have a sound understanding of the notion of hundredths. Whilst there are other cogent reasons for introducing the decimal-fraction recording prior to the common-fraction recording, there remains an inherent difficulty in the recording of decimal fractions and per cents, namely, the absence of a visual representation of the whole partitioned into a number of equal parts (i.e., the denominator).

The implication for probability is that only the common-fraction recording indicates the exact total number of outcomes in a given sample space. For example, given that the probability of drawing, without looking, a yellow marble from a container of marbles is $^{2}/_{5}$, students should be able to construct a mental representation of the problem more easily and accurately than they would if given the probability as 0.4 or as 40% simply because the common-fraction recording provides an unambiguous representation of both the number of favourable outcomes and the total number of outcomes.

Nik Pa (1989) identified several cognitive processes that children use in interpreting a given fraction such as $^2/_5$ but the expert cognitive structure appears to incorporate the mental partitioning of a whole into five equal pieces and the mental selection of two of those equal pieces. Conversely, when required to name a particular fraction represented by one of the models, the expert would first consider whether the whole had been partitioned into a number of equal pieces, would count the number of equal pieces to determine the name of the fraction and would then count the number of equal pieces to be considered in order to ascertain the size of the fraction.

When a group of Years 3, 4 and 5 children were asked to show 1 half with discrete objects, some used what Nik Pa classified as a "one-to-two comparison" (p. 18) in which the children made two parts (one part with one object and the other part with two objects. Nik Pa suggested that these children connected the meaning of a half with two parts but, although they may have been aware of the need to have equal parts, there was no concern for equality when they applied the comparing operation. Nik Pa stated that this scheme was the beginning of a ratio interpretation of fractional situations.

This embryonic ratio schema was again used when the fraction was $^1/_3$ and $^2/_3$. Nik Pa said that the child using the "one-to-many or many-to-many comparison" (p. 19) connected the meaning of $^a/_b$ to comparing a items with b items and showed no concern for the equality of the parts nor, this paper suggests, for the number of equal parts. Nik Pa stated unequivocally that this was a ratio schema.

"A ratio is a statement of comparison between, or a statement about, the relative sizes of two numbers or quantities" (Department of Education, 1990, p. 157). Hart (1988, p. 198) defines a ratio as "a statement of the numeric relationship between two entities" and proportion as involving "the equivalence of two ratios".

The main concept underlying ratios is a *part/part* relationship (Nik Pa, 1989). The cognitive structures required for interpreting a given ratio such as 2 : 5 is to think of 2 objects of one sort (e.g., triangles) and 5 objects of another sort (e.g., squares). Conversely, to determine a ratio when given a set of objects, the student need s only to count the number of objects of each type. For example, if given a set of fruit comprising 3 bananas and 4 apples and asked to describe the relationship of the bananas to the apples, the student merely counts the number of each type of fruit. When given a set of objects (e.g., 7 circles) and asked to place them in the ratio of 3 : 4, the student needs simply to partition the set so that there are 3 circles in one subset and 4 circles in the other. It should be noted, that unlike fractions, the whole does not have to be partitioned into equal parts.

Modern curricula exhibit a shift in pedagogical philosophy from knowing "how to do" mathematics to "understanding" mathematics and this has had enormous implications for the classroom teacher. If mathematical competence is believed to consist of knowing sets of correct procedures, then learning these procedures is simply a matter of watching, listening, practising and remembering, while teaching is showing, telling, providing many practice examples and, finally, testing to find out which students can carry out the procedures correctly (Lampert, 1986). Under this pedagogical philosophy, understanding appears to be equated with remembering rules and procedures and assessment is concerned with performance / the child's ordinary behaviour on a particular occasion (Ginsburg, 1981). If mathematical understanding is equated with performance then, according to Confrey (1987), it will be useless away from schools.

However, if mathematics education is believed to consist of acquiring an understanding of mathematical concepts and underlying principles, then teaching, learning and assessing become much more complex. Under this pedagogical philosophy, both teacher and learner share an active role in helping the student construct his or her own knowledge and in making sense of the knowledge thus constructed. Teaching and learning become a two-way interactive process and could best be described as "sense-making" (Lampert, 1986, p. 340) while assessment is concerned with competence / the child's highest ability at his or her current stage of development (Ginsburg, 1981) / rather than performance.

In an endeavour to understand what "understanding mathematics" means to a student who is being taught a new aspect of mathematics, most recent studies have taken the cognitive perspective that knowledge "**is** the cognitive structures of the individual knower and to know and understand mathematics from this perspective means having acquired or constructed appropriate knowledge structures" (Putnam, Lampert & Peterson, 1990, p.67). Leinhardt (1988) hypothesised that there were four types of knowledge / what students know before instruction (*intuitive knowledge*), what they acquired during instruction (*concrete knowledge, computational knowledge)* and what they acquire after instruction (*principled conceptual knowledge*) / and equated understanding with making connections between the knowledge types.

Whilst there were discrepancies in what researchers considered to be the distinguishing features of intuitive knowledge, all agreed that it was not derived from direct instruction (Baturo, 1992). However, concrete knowledge is universally understood to mean the knowledge gained through using manipulative material which can be *real-world* (e.g., coins, dice, biscuits), *representational* (e.g., counters, Unifix cubes, bundling sticks, base-10 blocks) or *pictorial*. It is generally held that proceeding from real-world to representational to pictorial material represents a natural sequence in promoting abstraction (Dienes, 1969; Bruner, 1966; Wilson, 1977).

Computational knowledge is the procedural knowledge required of the formal algorithms (e.g. addition, subtraction, multiplication and division) and the application of various formulae. Computational knowledge is primarily numerical (i.e., symbolic) and it constitutes the major part of the traditional school mathematics curriculum / "it is the goal of most teachers' instruction and it is what most achievement tests examine" (Leinhardt, 1988, pp. 121-122). Although failure to compute is almost always attributed to a lack of understanding, paradoxically, the presence of computational skill does not guarantee understanding (Erlwanger, 1975; Leinhardt, 1988).

According to Leinhardt (1988, p. 120), principled conceptual knowledge is the underlying knowledge of mathematics (e.g., the associative, distributive and commutative laws) from which the computational procedures and constraints can be deduced but which is generally not taught directly.

A study (Baturo, 1992) was undertaken to explore Year 12 students' cognitive functioning in the domain of probability in order to determine: (1) how students come to know a particular piece of mathematics curriculum, and (2) what knowledge must be constructed in order to know that piece of curriculum. To this end, Leinhardt's (1988) model of the knowledge types was used as a framework for analysing that understanding.

This paper reports on the findings that related to students' misconceptions, whether natural (intuitive knowledge) or generated (principled conceptual knowledge) in the domain of probability and, to this end, will focus on: (1) exploring Year 12 students' cognitive functioning in the domain of probability, (2) examining Year 12 students' intuitive, concrete, computational and principled conceptual knowledge of probability, and (3) identifying relationships between these knowledge types and the students' knowledge in the domains of fractions and ratios. Each of these foci will be addressed in the Discussion session. In conclusion, implications for teaching will be drawn with particular reference to the role of the affective domain in getting to know probability, the role of concrete material in promoting probability understanding, the effects of operating within different probability types and the effect of incorrect fraction and/or ratio knowledge on students' acquisition of concrete, computational and principled conceptual knowledge.

METHOD

A total of 31 Year 12 students participated in the studies overall. The clinical interview (Ginsburg, 1992; Ginsburg, Kossan, Schwartz, & Swanson, 1983) was the research methodology adopted, with each student being interviewed individually as he or she worked on a set of tasks developed for the study.

Subjects

Year 12 students were chosen because, essentially, the students required for this study were expected to exhibit more "expert" than "novice" intellectual functioning (Schoenfeld, 1983) in the domain of probability in order to analyse their understanding in terms of Leinhardt's four knowledge types. At the time of the study, secondary school students only (Years 8-12) had been exposed to any formal instruction in probability.

The students were from a private girls' school, a private boys' school and a state (public) coeducational high school. The Subject Masters at each school selected high and low achieving students within the Mathematics 1 course and the Mathematics in Society course. (In Queensland, entry into these courses is determined by the students' results at the end of Year 10. Only those students who are deemed capable of "handling" tertiary mathematics are admitted into the Mathematics 1 course.) Altogether there were 15 Mathematics 1 students (8 females – 4 high achieving, 3 low achieving; 7 males – 4 high, 3 low) and 16 Mathematics in Society students

(8 females – 4 high, 4 low; 8 males – 4 high, 4 low). Table 1 shows the mathematics course and achievement classification of the participating students and denotes the type of school attended.

Table 1

Categorisation of students' mathematics course, achievement rating and school type.

	COURSE AND ACHIEVEMENT			
	Mathematics 1		**Mathematics in Society**	
	High	Low	High	Low
Female	Andrea (G) Michelle (G) Sheila (S) Elise (S)	Camille (G) Karoline (G) Penny (S) Dianna (S)	Dene (G) Sarah (G) Jane (G) Karen (S)	Ali (G) Kerri (G) Monica (S) Marney (G)
Male	Eddy (B) Matthew O. (B) Scott (S) Paul (S)	Ben (B) Matthew G. Douglas (S)	Nicholas (B) John (B) Michael (B) Jason (S)	Joe (B) Brendan (B) Damien (S) Trent (S)

Note. (G) indicates the girls' school; (B) indicates the boys' school; (S) indicates the state school.

Instrument

The interview instrument consisted of a questionnaire designed to elicit the students' feelings about probability and the activities they had undertaken as part of their formal study, and 19 tasks. This paper will report on those tasks that yielded most insight into the students' thinking (Tasks 1A, 1B, 3, 4, 10, 14, 15, 18).

When developing the tasks, the major concern was that they should reveal the students' cognitive processes related to elementary notions of probability. Therefore, all of the tasks were limited to: (1) considering a single sample space and the associated possible outcomes, (2) comparing different events in a single sample space, (3) comparing the same event across several sample spaces all of which had the same numbers of total outcomes, and (4) comparing the same event across several sample spaces all of which had different numbers of total outcomes. Single-stage tasks only were used so that the students did not have the extra complication of worrying about replacement and thus, hopefully, eliminating a potential source of ambiguity. Tasks 3, 4 and 10 were based on the tasks Jones (1974) designed for his doctoral study.

Leinhardt's model of the types of knowledge that students acquire as they come to know a piece of curriculum was a major theoretical construct underlying the development of the tasks as the model was to provide the framework for the analysis of the students' cognitive processes. Therefore, the task items were developed to reveal the students' intuitive, concrete, procedural and principled conceptual knowledge of the elementary probability notions. It was difficult to know, beforehand, what tasks would be novel and thus provoke intuitive knowledge but it was anticipated that the students' initial responses to any task would probably give a reasonable indication of their intuitive knowledge.

The use of language was another consideration in the development of the tasks. Probability is very rich in informal language (e.g., likelihood, chances) as well as formal language (e.g., probability). It was decided to use the formal language in the tasks but informal language in any contingent explanations. Another problem occurring through the language was that the task directions were very likely to lead the thinker in a particular direction. For example, Task 3 was initially to duplicate one of Jones' (1974) tasks which involved a container of coloured balls in which 2 red balls, 3 blue balls and 1 yellow ball were clearly visible. The question asked by Jones (1974, p. 94) was: *If a ball is drawn from the container, what colour is it most likely to be?*

However, the question assumes that there **is** one colour that would most likely be drawn and thus leads the student to look for the largest number. Moreover, there is nothing to say that the ball is not to be deliberately selected. To accommodate these language problems, the questions in Task 3 of this study were changed to: *Does each colour have the same chance of being drawn from this container? Why? If you were to draw, without looking, a marble from this container, which colour would you be most likely to draw?*

Finally, the tasks were to incorporate a variety of probability materials such as spinners, dice, marbles and so on in order to accommodate Diene's (1969) *Mathematical Variability Principle* which postulates that students show understanding of a concept if they are able to abstract the essential mathematics from a variety of materials relevant to the concept.

Procedure

Each student was interviewed individually and the tasks were presented in the same order. The tasks were read to the students and contingent explanations given when it was deemed to be appropriate. Each interview took approximately 40 minutes and was video-taped

Analysis

The video tapes were transcribed into protocols and, from these protocols, inferences concerning the students' knowledge of probability were drawn. Where appropriate, these inferences were categorised in terms of Leinhardt's knowledge types.

RESULTS

Each task that is pertinent to this report is presented with the directions, a description of the materials, the objective/s, the results with protocols and inferences as to the behaviours noted.

TASK 1A

Directions:

Objective: To evaluate the student's intuitive understanding of fair and unfair outcomes.

Materials: 12 discs, each with a diameter of 20 cm, were used to represent spinners. (A spinner is sometimes used in games of chance and can have colours or numbers.) For this task, eight of the discs were "fair" spinners because each outcome (a colour) had an equal chance of occurring and four were "unfair" because each outcome did not have an equal chance of occurring. Each student was presented with the 12 discs to physically manipulate.

Results: No student sorted correctly on the first attempt and only 1 student (a high achieving Mathematics 1 student) was successful on the second attempt. This task was changed to Task 1B below after 15 students were interviewed because it was quite obvious that they were unable to overcome their lack of concrete experiences with this type of probability material.)

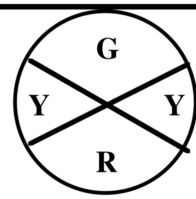

Figure 2.
Spinner in Task 1B.

Task 1B

Directions:

Objective: (1) To determine whether the students displayed any conflict between visual perception (the amounts of colour don't look equal) and cognition (knowing that if the two yellow parts were adjacent, they would cover the same amount of area as the red or the green); and (2) To determine which form of processing information (visual or cognitive) the student's preferred.

Results: Eleven of the 16 students were unequivocal in their responses (6 said that it was a fair spinner which was inferred as indicating cognitive processing whilst 5 said that it wasn't fair which was inferred as indicating perceptual processing. When these students were asked which colour would be more likely to be spun, they either thought yellow because it had "2 chances" or they thought red or green because either had "more" than yellow. Five of the 16 students (including 2 high achieving male Mathematics 1) vacillated as indicated by the following protocol.

Eddy: *No. Maybe because these two [yellow] are right opposite so these two [red, green] would have more chances. But if these two [yellow] joined together are the same as these two [red, green] then it would be a fair spinner . . . but these two [red, green] would have more chances.*

TASK 3

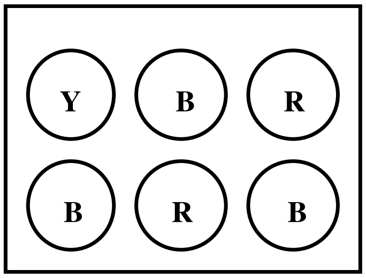

Figure 3.
Marbles in Task 3.

Directions:
Why? What would be the probability of drawing a blue marble? a red marble? a yellow marble? Write the probabilities.

Objective: To evaluate the students' ability to compare the probability of two or more events in one sample space; (2) To determine the students' natural language for describing probabilities and their preferred style for writing probabilities.

Results: All students could compare the probabilities but they used a mixture of fraction and ratio language (e.g., 1 over 4; 1 in 4) to describe the probabilities and a mixture of fraction and ratio recordings for the probabilities. The two students who attempted to use the per cent language and recording did so throughout the tasks with very poor results. This task was included for comment in this paper because, when looked at in conjunction with the other comparison tasks, it showed an insight into how the students changed their comparing strategies to suit the particular task. This will be commented on in the Discussion section.

TASK 4:

Directions:

Objective:

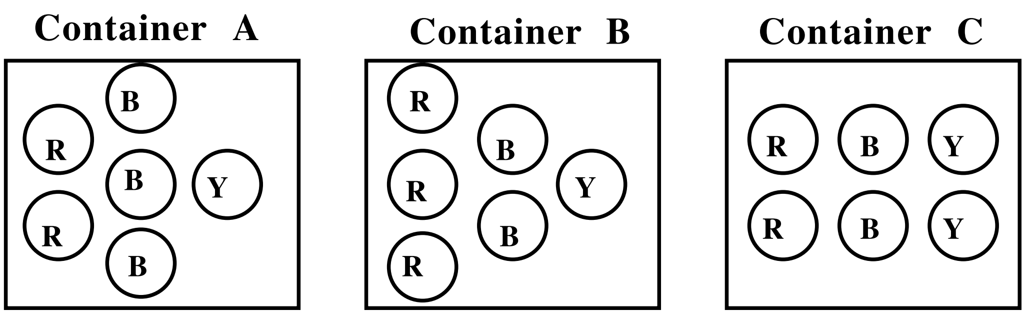

Figure 4.

Marbles used in Tasks 4 and 10.

Results: Every student answered correctly but, as the protocols reveal, the explanations revealed that the students used fraction cognitive processing, ratio cognitive processing or whole-number processing.

Michelle: *You've got 3 in 6 chances from Container A and only 2 in 6 chances in Container B so A gives you a better chance because it's half a chance as opposed to a third.*

Eddy: *Because this one's [Container A] got only 50% chance and this one's[Container B] only got 2 thirds which is less than 50%.*

Matthew: *They've both got the same amount in each [considering the total number of equal parts] but there's more blue in Container A.*

Karoline: *No, because there's 3 in Container A and 2 in Container B.*

Ben: *No, because with Container A more marbles are blue.*

The responses given by Michelle, Eddy and Matthew (all M1/H) were deemed to be reflective of fraction cognitive processing whilst the other responses were deemed to reflect the comparison of numbers or ratio cognitive processing. As well, the first three responses have been ordered in decreasing levels of sophistication, judged solely on the quality of the language used to describe the processing. The fact that Eddy made the same error in considering 2 out of 6 as 2 thirds as he did in Task 3 was not considered to detract from the quality of the cognitive processing he employed. The last two responses which were considered to reflect comparison thinking were also ordered in decreasing levels of sophistication, with the criterion for judgment being the use of explicit numbers instead a nebulous "more".

TASK 8:

Directions:

Objective: To determine the students' understanding of a certain event.

Results: Eleven of the 31 students responded incorrectly with 10 of them being Mathematics in Society students. Their explanations revealed that, to them, *possible* and *certain* were synonymous and the most common explanation given was: *because there's at least one yellow marble in each container*. Whilst most of these students were guided to the correct response, their voices and body language were interpreted as indicating a lack of any real belief in their changed answer.

TASK 10:

Directions:

Objective: To evaluate the students' ability to compare the probability of an event across three sample spaces when the total number of outcomes is different.

Results: Of the seven students who did not respond correctly, five focused on the number of favourable outcomes only (e.g., *Yes, because there's 2 yellows in each container*) while the other two gave explanations that were unclear (see the following protocols).

Michael: *No, because there's more blue marbles than yellow ones.*

Dene: *Yes, but no . . . because there's 3 blue ones in Container A so you'd still be fighting the blue ones.*

TASK 13:

Directions:

P (yellow) = $^3/_8$; P (blue) = $^1/_2$; P (green) = $^1/_8$

Objective: To evaluate the students' understanding of the elementary probability notions by having them construct a sample space to match given symbolic probabilities. (In the previous tasks, the sample spaces were given and the students were required to give the probabilities.)

Results: Twenty-two of the students could do this without any problem. Of the remaining nine students, six put out 3 red, 1 blue, 1 green (matching the numerators). One student put out 2R, 3B, 1G because "you've been using 6 constantly" (referring to the numbers of marbles in the previous tasks). One student put out 8R, 2B, 8G (looking at the denominators). The remaining student put out 3R, 2B,1G, 2O (orange) so that she would have 8 counters altogether. When probed, she revealed that she hadn't realised that the given probabilities were all that could be ascertained from the sample space. Therefore, her construction was technically correct except for the 2B and when asked what the probability would be of getting blue, she replied: *2 in . . . isn't it? I don't know*.

TASK 14

Directions:

P (yellow) = $^1/_3$; P (pink) = $^2/_9$; P (purple) = ...

Objective: To evaluate the students' intuitive understanding that the sum of the probabilities in an experiment is 1.

Results: All of the students realised that, to find the missing probability, the two given probabilities had to be added and then subtracted from 1. When asked why they took $^5/_9$ from 1, the replies of several students indicated that they understood that the probability of all the outcomes in an experiment could not exceed 1. Moreover, their overt behaviours (tone of voice, together with the surprised look they directed at the interviewer) was interpreted as indicating that they thought this was a trivial question, the answer to which being common knowledge.

Andrea: *Because these (indicating the given probabilities) are fractions of 1 whole (indicating a circle with her fingers) so I just subtracted the addition of those two from 1.*

Michelle: *Obviously all three numbers (probabilities) had to be used to make a total of 1 (a whole), so I took the 5 ninths away from the whole and got 4 ninths.*

Matt. G: *You need to make up the rest to get a whole probability so you need to make it 9 over 9 and then take away the 5 ninths.*

Camille: *Because 1 is the highest probability you can get.*

Ben: *The total probability has to be 100% so the three colours is the whole.*

John: *Well, I was working on a whole amount that had 9 parts*

Sarah: *Because you're trying to find out an "x" so to do that you have to take it away from 1 because, in probability, 1 is a very important number. I just can't remember the formula.*

However, 7 of the 31 students could not operate with common fractions. Three students added the numerators to get $^3/_{12}$. Joe renamed $^1/_3$ as $^2/_9$ giving him $^4/_9$ which he promptly renamed as $^2/_3$ while it appeared that Ali had a much more fundamental problem with renaming common fractions as her protocol revealed.

Ali: *The probability of yellow was 1 third (to herself). So you'd have 9 . . . no, you wouldn't .. .yes, you would. So that means you'd have spun it 9 times . . . does it?. . . Actually, I don't think I know about this. . . . So does that mean we're saying that there is 1 chance out of 3 chances of getting yellow? But how come you've only got 3 chances to get a yellow when you've got 9 chances to get the pink? That's what I don't understand!*

Two students exhibited an interesting behaviour that appeared to be task-specific in that the fractions used seemed to promote thinking that was misleading. The numerators (1, 2) could have been thought of as a counting sequence so that the next numerator would be 3. The denominators could also have been thought of as a number sequence (3 x 3 = 9; 9 by 3 = 27) so the missing fraction was thought to be $^3/_{27}$. That is, both of these students seemed to interpret the given probabilities as comprising a pattern. Their protocols support this assumption.

Karoline: *1 is to 3, 2 is to 9 so 3 is to 27. Hang on a minute* . [I: What picture do you have in your mind?] *I'm thinking 1, 2, 3 . . . it's going up*.

Kerri: *A third, 2 ninths . . . this is just like algebra . . . 3, um , 3 over . . . 3 twenty-sevenths. Well, you've got 3 parts so you times your bottom by 3 which gives 9, plus you keep adding on because you've got 3 parts so that's the first part($^1/_3$), second part pink, third part purple so it'll be 3 over and you've got 3 parts altogether so multiply what you've got on the bottom by 3 which gives you 9 (pointing to pink) and then you multiply 9 by 3 (pointing to purple) which gives 27.* [I: Why do you have to multiply 9 by 3?] *Because you've got 3 parts . . . I don't know (laughing). Yes, but I could also do it another way. It could be . . . 3 fifteenths.*

TASK 15:

Directions:

15.1 Read each probability.

15.2 Do all of the probabilities make sense? [If "yes", ask the students to describe or construct an experiment to match P = 2.]

15.3 Are there any probabilities that have the same value? If so, which ones?

15.4 What does P = 0 mean? [Ask for an example.] What is this type of event called?

15.5 What does P = 1 mean? [Ask for an example.] What is this type of event called?

Objective:

Results: All of the students could read the probabilities and all but one could identify those that had the same value. Only 13 of the 31 students said that P = 2 was an "odd" probability and all of the Mathematics 1 students in this group could justify their response in mathematical terms (e.g., *1 was the most you could have*; *P = 2., because that's 200% and, usually, the maximum is 100%)*. The remaining students who responded correctly could only give subjective/intuitive explanations such as: *It just seems out; it just seems strange*.

Four students thought that P = 0 seemed odd. As Jane explained: *on most occasions, the probability of getting something is above zero*. Two students thought that both 1 and 2 seemed odd whilst two other students thought that 25% seemed odd because they had never seen a probability written as a per cent. The remaining ten students didn't think any of the probabilities were odd.

All of the students could construct an experiment to show P = 0 but only six students could name it correctly as an impossible event (with two students admitting they guessed). Only a few others attempted to name the event and used the following terms: *ungettable, null, non-event, zilch*.

Fifteen students could construct, without ambiguity (e.g., 3 R), an experiment in which the probability of an event occurring was 1. Ten other students were technically correct in that each put out one red counter and said that the probability of drawing red was 1. However, when asked if they could show P = 1 with more than one counter, only three of these students unhesitatingly put out more counters of the same colour. Two of the other students put out 2 blue counters but when asked what the probability of getting red was, each quickly amended the sample space by changing the two blue counters to two red counters. Of the remaining five students who had originally constructed a sample space with just one red counter, the protocols revealed that they either focused on the number of favourable outcomes without considering the total number of outcomes or they ignored the number of favourable outcomes and focused instead on the individual trial. The following protocols highlight these behaviours.

Nick: [Considering 2 red counters] *Then the probability of getting red would be 2.*

Karen: [Considering 3 red counters] *The probability of getting red is 3.*

Matthew O.:

Jane: [Considering 4 red counters] *The probability of getting red would be a quarter.*

Monica: [Considering 3 red counters] *The probability of getting red is 1 third.*

Some students confused a certain event with a possible event. For example, Joe (MS/L) had put out 1 blue counter and 2 yellow counters (1B, 2Y) and said that the probability of drawing blue was 1. Matthew G. (M1/L) put out 5 blue, 4 red and 1 yellow and proceeded to "validate" his response by writing the probabilities in symbolic form $^5/_{10}$, $^4/_{10}$, $^1/_{10}$). He completely failed to see any discrepancy in P = 1 and P = $^1/_{10}$. Kerri (MS/L) put out 2R, 2B, 2Y and then said: *If you've got 1 or more . . . as long as you've got 1, then you've got every chance of getting a red, blue or yellow.* The following two students sounded quite confused.

Eddy: *I just can't find P is 1 because you haven't got a percentage. Like you haven't got any other thing to compare with 1 so therefore it can't be . . . If I got 1 here (reaching for 1 red counter), the probability is 100% but if I've got a few others here (indicating 4 blue counters), it'd be less.*

John: *See that's the difference [from P = 0] . . . see, if you're working on the possibility of 1 and the possibility is point two five, you'd say that, because this is point two five, this (P = 1) would be a hundred, right? So they'd be all red (showing 4 red counters). Then, if you're saying the possibility is 1, you might have 99 red and 1 / sorry, the other way round / 99 blue and 1 red so then you'd be talking in these [P = 1] . . . in numerals and not per cent. So you'd say that the possibility of getting blue is 99 in a hundred and 1 chance of getting red so it depends if you're talking in per cent or if you're talking in numerals.*

TASK 18

Directions:

2 : 5 **$^2/_5$**

Objective: To determine whether the students' cognition of ratio is based on a *part/part* schema or a *part/whole* schema.

Results: This task was included after the first three students were interviewed as their protocols had revealed that there may have been some confusion between a ratio cognitive structure and a fraction cognitive structure. These results, then, were obtained from 28 of the 31 students. An interesting phenomenon was revealed by these results, *viz*., that the responses were school specific. Therefore, the schools attended are given with the results. The girls' school is indicated by G,. the boys' school by B and the state school (coeducational) by S.

Nearly all of the students read the ratio as "two is to five" but Kerri (G) seemed to be confused, at first, between the decimal point and the colon of the ratio as she began to read the ratio as "*2 point*" and then followed it with "*2 to the ratio of 5*". All students from the girls' and boys' schools read the fraction formally (i.e., as "*2 fifths*") but several of the state school students read it more informally as "*2 out of 5*" or "*2 over 5*". None, however, read it as a ratio (as they had sometimes been doing during the interviews).

Only those responses that could be validated were considered to be correct. That is, if a student claimed that the ratio and the common fraction had the same meaning and could demonstrate the claim with concrete material or diagrams, then that student's response was taken as being a correct response. Similarly, if a student claimed that the ratio and the common fraction had different meanings and could validate the claim, that student's response was also taken as being a correct response. Table 2 summarises the students' correct and incorrect responses and the schools they attended.

As can be seen from the table, 18 out of the 28 students gave incorrect responses with the most common incorrect response (i.e., inability to validate the response) being that the fraction and the ratio had the same value. All of the state school students belonged to this category thus indicating that this phenomenon appears to be school specific.

Table 2

Categorisation of students' responses for Task 18.

	Same Meaning		**Different Meaning**	
Correct	Karoline (G) Nicholas (B)	Kerri (G)	Andrea (G) Sarah (G) Matthew G (B) Brendan (B)	Camille (G) Jane (G) John (B)
Incorrect	Michelle (G) Elise (S) Penny (S) Karen (S) Matthew O. (B) Scott (S) Paul (S) Jason (S)	Marney (G) Sheila (S) Dianna (S) Monica (S) Joe (B) Douglas (S) Damien (S) Trent (S)	Eddy (B)	Ben (B)

In the *Same meaning/Correct* category, Nicholas and Kerri each put out 2 red counters and 3 blue counters to represent both the ratio and the common fraction so it seems as though Nicholas and Kerri use a *part/whole* schema for both ratios and common fractions. On the other hand, Karoline seems to have a ratio schema that enables her to see it as *part/part* and *part/whole* because she initially put out 2 yellow counters and 5 blue counters to represent the ratio and then realised that this wouldn't represent the fraction. She then changed her representation to 5 yellow counters, indicating 2 of the counters as representing 2 fifths of the whole set of counters and saying, *2 part yellow, 5 part the whole,* to represent the ratio.

In the *Same meaning/Incorrect category*, the most common behaviour was to put out 2 red counters (for example) to represent the first term of the ratio and then to put out 5 different coloured counters to represent the second term. On the whole, most of the students represented the ratio very confidently and quickly. However, when asked how this represented the fraction, $^{2}/_{5}$, many students realised that they could only have 5 counters altogether and the look of surprise when they realised that they didn't really see the ratio and the fraction as having the same value indicated that this was probably the first time they had ever had to validate their belief.

On questioning these students as to why they had been so sure that the ratio and the fraction had the same value, they admitted that they had been told by their teachers and, in fact, had been encouraged to convert ratios to common fractions whenever possible. The students had never thought that it was necessary to validate this procedure. This shook the confidence of several of the students, particularly Scott (a very high-achieving Mathematics 1 student) who had been so quietly sure of his knowledge (and with good reason) throughout the interview. This typified the behaviour of the state school students.

Whereas the previous students had constructed the ratio first, Michelle constructed the fraction first. She put out 2 red and 3 blue counters but when asked where the "5 part" was, she became confused and quickly put out 2 more blue counters so that she had 2 red and 5 blue counters. However, Michelle could not then accept this as a representation of the fraction. She admitted that she had been taught that "*those dots* [in the colon] *mean the same as that* [the bar line in the fraction]" and had never been shown any concrete or pictorial representation of either a ratio or a common fraction.

Other behaviours were also noted. For example, two students merely held 2 red counters above 5 blue counters to replicate the way the common fraction terms were written. Matthew O. put out 2 red counters, 2 yellow counters and 1 blue counter to represent the fraction saying that "*2 fifths* [of the set of counters]*were red*" but then he could not relate this to the ratio. He finally said he'd have to add 2 more counters to represent the ratio. It was obvious from Matthew's facial expressions that this was a revelation to him.

All of the students in the *Different meaning/Correct* category represented the ratio by using 2 counters of one colour to represent the first term and 5 counters of a different colour to represent the second term. Each student then took away two of the 5 counters to show the fraction. These students seem to have a *part/part* schema for ratios and a *part/whole* schema for common fractions.

In the *Different meaning/Incorrect* category, Eddy and Ben both said that the ratio and the fraction did not have the same meaning. Ben, however, was not too sure because "we don't do much ratio". He represented the fraction with 2 red counters and 3 blue counters and said that red was 2 fifths but then he said "*2 to 5*" and this must not have appeared correct to him because he changed his mind and added another 2 blue counters. When asked what fraction was represented, he pointed to the red counters and said, "*That's 2 sevenths of the total*".

Eddy proceeded to represent both the fraction and the ratio with 2 red counters and 3 blue counters. However, his representations merely replicated the written representations of these two symbols. For example, to show the fraction, he made two rows of counters placing the row of 2 red counters above the row of 3 blue counters; to show the ratio, he made two columns of counters with the column of 2 red counters in front of the column of 3 blue counters.

DISCUSSION

As stated earlier, this paper set out to report on those misconceptions that were revealed in Baturo's 1992 study. The following discussion will look at each of the foci in turn.

Year 12 students' cognitive functioning in the domain of probability

The results indicated that the students performed quite well overall. However, the protocols revealed that they used a variety of cognitive schema when contemplating probabilistic situations. For example, there appeared to be evidence that the students were using the *part/whole* fraction schema, the *part/part* ratio schema and the *whole/whole* comparison schema. As well, there appeared to be evidence that many of the students did not employ just one schema but rather called on a medley of schema, depending on the type of problem/situation. Karoline's responses to Tasks 3, 4 and 10 (see Figures 2 and 3) illustrate this point.

Task 3: *No; because there's 2 yellows, 3 blues and 2 reds*

Task 4: *No; because there's 3 [blue marbles] in Container A.*

Task 10: *No; because there's another . . . like there's 2 sixths [in Container C] gives a better chance than 2 sevenths [in Container A].*

Karoline's responses to Tasks 3 and 4 appeared to reflect a comparison schema whilst her response to Task 10 seemed to indicate a fraction schema. To understand Karoline's shift in cognitive schema, an analysis of the tasks reveals the ways in which they may be different. In Task 3, only one sample space was given so Karoline could have taken either one of the solution paths shown in Figure 6.

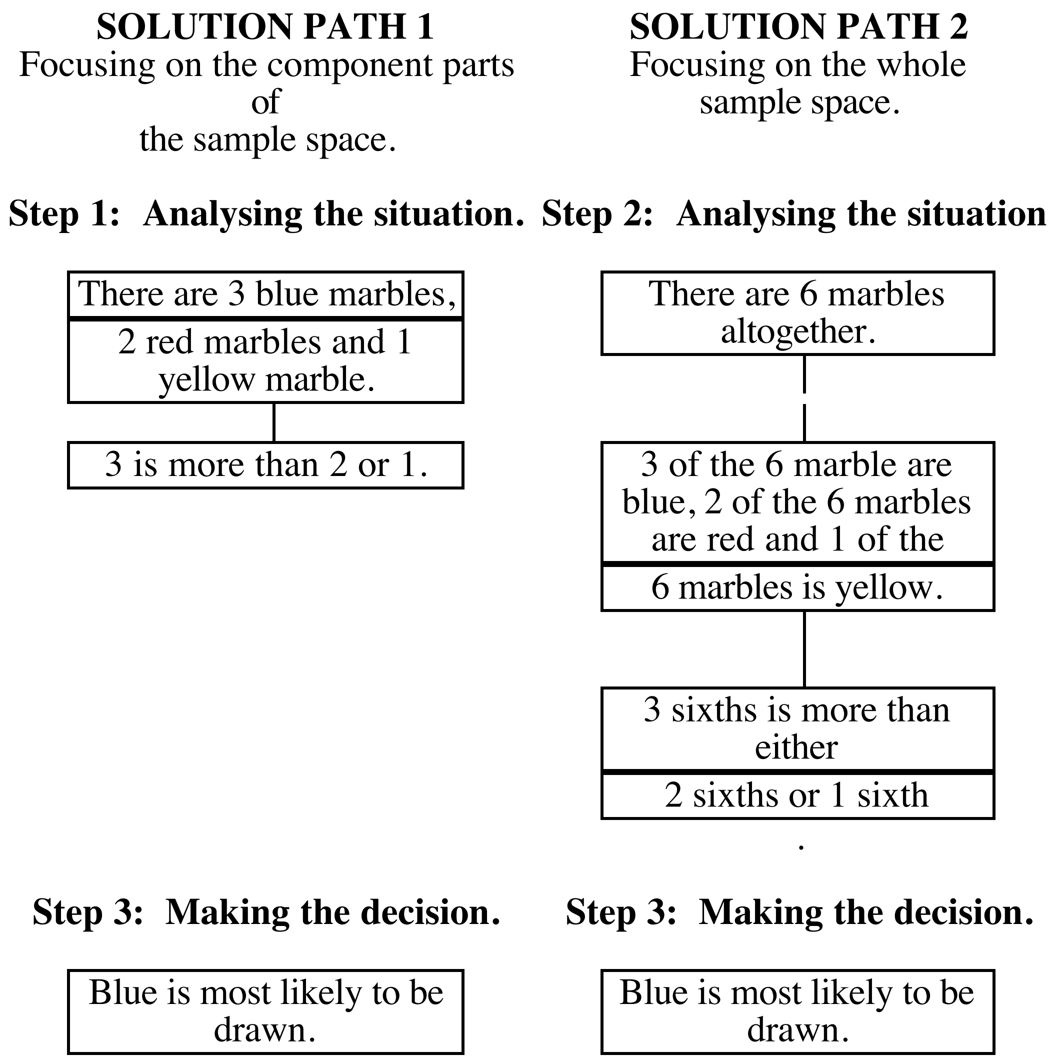

Figure 6. Possible solution paths for Task 3.

In Tasks 4 and 10, the students were required to compare the probability of an event across two sample spaces / the sample spaces in Task 4 had the same number of total possible outcomes (see Figure 7) whereas the sample spaces in Task 10 had different numbers of total possible outcomes (see Figure 8). However, whilst Tasks 3 and 4 could be solved successfully by simply comparing the number of favourable outcomes (the numerators) as though they were whole numbers, Task 10 could not be solved successfully without a consideration of the favourable outcomes in terms of the total possible outcomes. Figure 9 suggests the solution paths that may be taken when two or more sample spaces are to be compared.

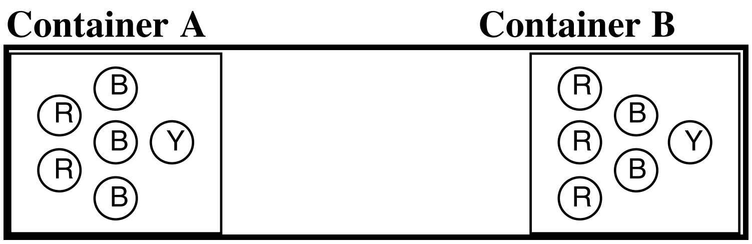

Figure 7. The sample spaces to be compared in Task 4.

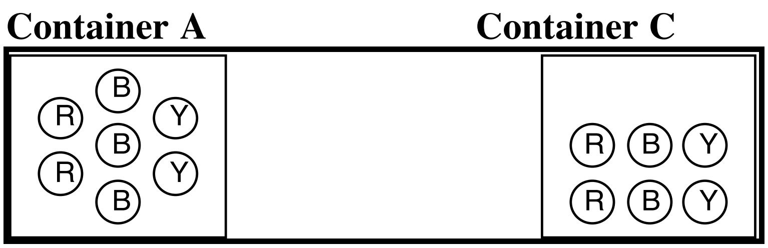

Figure 8. The sample spaces to be compared in Task 10.

As can be seen from Figure 9, both solution paths for Task 4 should produce the correct answer. That is, when the sample spaces have the same number of possible outcomes, both the ratio/comparison schema (Path 1) and the fraction schema (Path 2) will produce the correct response. However, when the sample spaces have a different number of possible outcomes, only the fraction schema (Path 4) will produce the correct answer.

The findings of this study tend to support Piaget and Inhelder's (1975) view that the notion of chance precedes true notions of probability and that these notions of probability cannot be fully established until the child has the necessary cognitive structures which, according to the findings of this study, are the common fraction schema. Fischbein (1975) and Jones (1979) posited that children as young as 5 years of age have the cognitive structures required for processing elementary probability notions. However, Fischbein used marbles only (the set model) in his study, thereby enabling his young subjects to employ a ratio/comparison strategy with success. Fischbein pointed out that the subjects "tended to estimate chance by using ratios" (1975, p.177) rather than by using common fractions which, he stated, was the mathematical schema underlying probability.

In his 1974 study, Jones used both spinners (area model) and marbles (set model) and found that set embodiments were more effective in promoting probabilistic thinking than were the measurement/area embodiments. However, this finding is challenged in light of the results of this present study in that probabilistic thinking seems to be inextricably entwined with common fraction schema, not with ratio/comparison thinking. The literature has also revealed that the measurement/area model appears to be superior to the set model in promoting common fraction conceptual knowledge.

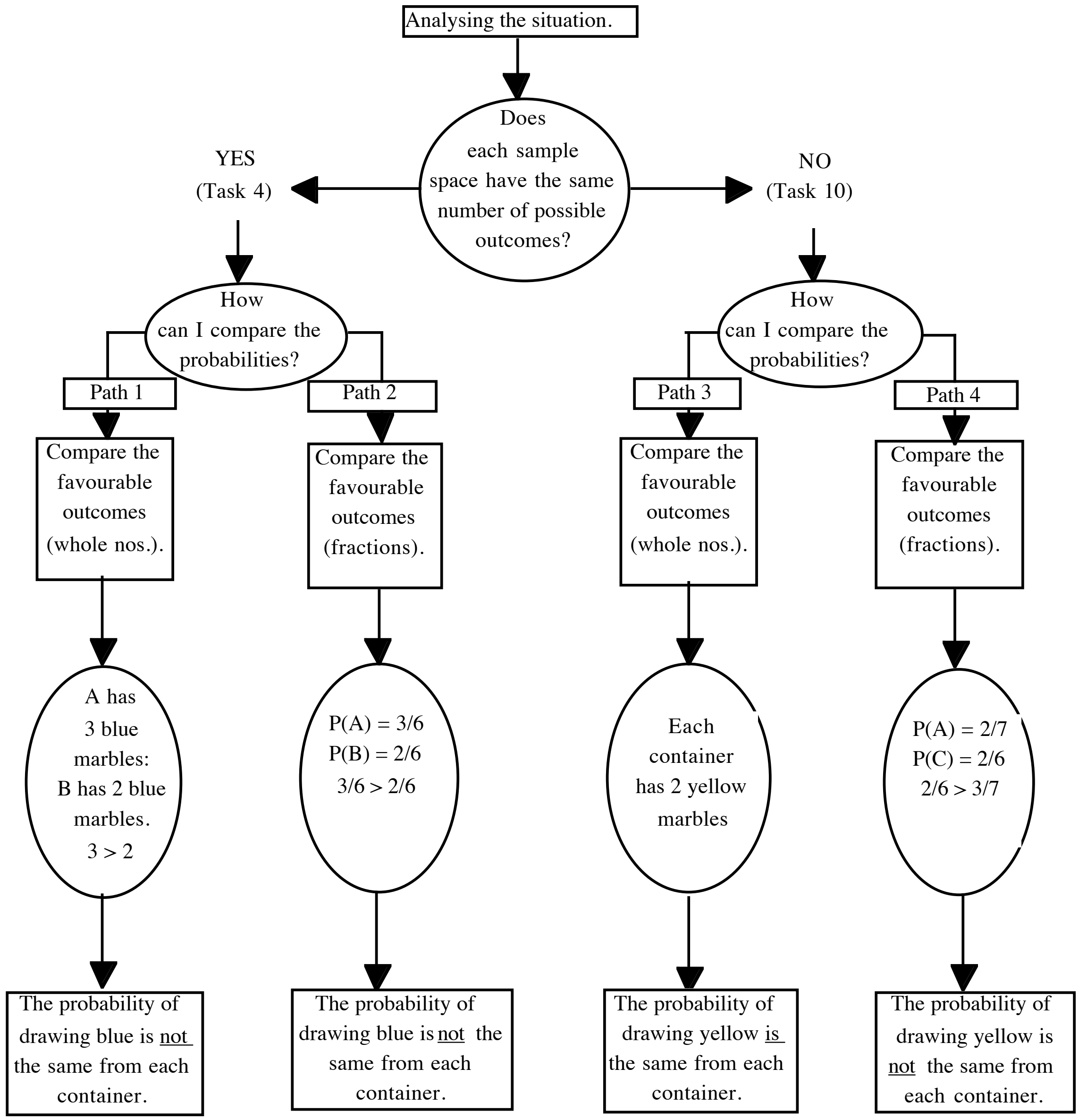

Figure 9. Possible solution paths for Tasks 4 and 10. Paths 1 and 2 indicate the solution paths for Task 4. Paths 3 and 4 indicate the solution paths for Task 10.

Year 12 students' intuitive, concrete, computational and principled conceptual knowledge of probability

Intuitive knowledge

A consideration of fairness/unfairness with respect to probabilistic events does not seem to form part of the students' intuitive knowledge of probability. Such a consideration appears to be fundamental to an understanding of probability as it encourages the students to move away from the perceptual (i.e., subjective) nature of probabilistic events to the analytic and quantitative (i.e., objective) aspect of measurement. That is, it appears to be the intuition that underlies their ability to move on from *estimating* chances to *measuring/calculating* probabilities, supporting Davies" (1965) differentiation between the intuition and the concept of probability.

When asked to compare the probability of an event occurring in one or more sample spaces, the natural/intuitive process employed by the students appeared to be to compare the number of favourable outcomes only. However, there is some doubt as to whether this is actually an intuitive process or merely an expedient process. That is, the students may well have access to more than one cognitive process and choose the process that is most expedient in answering the task.

There appears to be two intuitions which need to be accommodated when comparing the probability of an event across two or more sample spaces (that the same number of favourable outcomes results in equal chances, and that an event has more chances of occurring in a smaller sample space. The studies showed that some students seemed not to have accommodated these two intuitions whilst some appeared to be in the process of accommodation of the two intuitions.

Many of the students exhibited strong intuitive knowledge that the numerical probability of any event ranged from 0 to 1 but they appeared not to have this knowledge in computational or principled conceptual form because many students accepted that P = 2 was a legitimate probability statement and proceeded to "demonstrate" its legitimacy with concrete material.

Concrete knowledge

The questionnaire revealed that almost all of the students who participated in this study had had virtually no "hands on" experiences with concrete material.

The students' concrete experiences, if any, had been limited to models that incorporated the set embodiment (e.g. marbles, dice, coins and cards). None of the students had had any experiences with models (either concrete or pictorial) incorporating the measurement/area embodiment (e.g. spinners).

Although the students, in the main, had had no concrete experiences, many showed an ability to both interpret and construct probability models. However, there appeared to be some evidence that the lack of concrete experiences appeared to be more of a problem for low-achieving students than for high-achieving students.

Computational knowledge

All of the students could read the symbolic probability notation and all could identify the probabilities that had the same value even though they were recorded in different fraction forms (common fraction, decimal fraction and per cent). However, very few could recognise the symbolic expression of an impossible event and a certain event.

Most of the students demonstrated an ability to compute with common fractions although several of these students described a probability in ratio terms. For example, a student may write a probability as $^1/_3$ but refer to it as 1 in 3.

The two students who preferred to work with per cents instead of common fractions were unable to explain their responses.

When asked whether 2 : 5 and $^2/_5$ represented the same amount, all of the state school students said that they did whereas only half of the private school students said that the ratio and the fraction represented the same amount. Very few of the 28 students interviewed could provide either a real-world situation or a concrete representation to support their belief.

Principled conceptual knowledge

The students in this study appeared to have an inadequate understanding of impossible and certain events, perhaps because their study of probability may have focused on possible events.

Some students had a limited, even confused, understanding of certain events For example, their responses indicated that they connected a certain event with the number, 1, but their concrete representation of a certain event revealed that this meant that there could only be one possible outcome in the sample space (e.g., 1 red marble). This may be the result of inadequate concrete knowledge supporting the computational knowledge.

Stemming from the behaviour in (2) was the apparent belief that P = 2 was mathematically legitimate. This cognitive phenomenon could be attributed to the erroneous principled conceptual knowledge concerning certain events. That is, P=2 may be seen as the result of two certain events, for example, the result of two trials each of which is a certain event. However, for some students, it was the probability obtained when the sample space had 2 outcomes both of which are the same (e.g., 2 red marbles).

Relationships between Leinhardt's knowledge types and Year 12 students' knowledge in the domains of fractions and ratios.

The study indicated that there may be a link between probability competence and fraction competence, particularly common fractions. Those students whose protocols indicated that they were using mainly *part/part* ratio or *whole/whole* comparison schema did not perform as well as those students whose protocols revealed that they were using the *part/whole* fraction schema. What was not clear from the protocols was whether the students who used the ratio or comparison schema did so from choice or from necessity. That is, did they also have well-developed principled conceptual fraction knowledge as well as principled conceptual ratio or comparison knowledge or was the ratio/comparison schema the only schema available to them? If the former was the true situation, then this would seem to indicate that these students intuit that probability is a matter of estimating rather than measuring chances. Some students, then, appear to consider the whole sample space as an entity whilst other students appear to focus on the component parts of a given sample space. Thus, there appears to be a desirable relationship between intuitive probability knowledge and common fraction knowledge.

This study found that the students do not appear to make connections between the four knowledge types. For example, having good intuitive knowledge of some aspect of probability did not necessarily mean that the student had developed sound computational or principled conceptual knowledge of that aspect of probability. As well, good computational knowledge often did not appear to reflect well-developed principled conceptual knowledge. Whilst computational knowledge alone may have been the desired outcome of pedagogical philosophies of the past, it is not the desired objective of the current philosophy. Rather, modern pedagogical philosophy seeks to promote understanding and this would appear to require harmony amongst the four knowledge types. What appears to be the major linking factors in promoting this harmony are effective mental representation and verbal explanation both of which seemed to be lacking from the students' repertoire of probabilistic experiences. Without these two factors, students are unable to validate their intuitive and computational knowledges. Concrete knowledge is deemed to be crucial in developing effective mental representation whilst the pertinent language is essential for verbal explanations which are the crux of communication between student and student, student and teacher.

The above statements can be exemplified by the students' behaviours in Tasks 14 and 15. For example, in Task 14, nearly every student intuitively knew that the sum of the probabilities of the events in a given sample space cannot exceed 1 and they used this knowledge without being consciously aware of it, when answering the task. As well, nearly every student could read the symbolic notation, $P = 1$, (Task 15) thus exhibiting computational knowledge of this probabilistic phenomenon. However, very few students could adequately demonstrate an example of $P = 1$ with concrete materials, thus exhibiting a lack of concrete knowledge in this area. Furthermore, the students, on the whole, could not describe this type of event using the correct probability terminology. The lack of correct terminology seriously limits the students' ability to link mathematical experiences to real-world events, thus prohibiting the student from making sense of what is learnt.

CONCLUSIONS

Many of the incorrect behaviours exhibited by the students in this study appeared to be teacher-generated either through inappropriate "recipes" (e.g., *the dots are the same as the line; a fraction is the same as a ratio*) or through neglecting to make connections between the knowledge types (as discussed in the previous section). As well, the students' instruction appeared to have focused on possible events with very little emphasis on impossible and certain events. From the students' reaction to the concrete material used in the study, it was inferred that most instruction was "one-way", that is, the students were given an experiment (usually in pictorial or language form) and asked to record the probability; "reverse" instruction, that is, being given the probability of an event and then asked to construct an experiment to meet this condition, appeared to be a novel situation for the students in this study.

Furthermore, the questionnaire that was administered in the wider study (Baturo, 1992), revealed that the students, on the whole, (1) could see no purpose in learning probability, (2) did not really enjoy nor understand the probability lessons they had had, and (3) did not have adequate concrete experiences in the course of learning. It could therefore be inferred from these findings that the students' probability education did not reflect modern pedagogical philosophy

Helping students make sense of what they are learning by establishing the purpose and/or usefulness of the topic and by giving real-world problems that are appropriate to their interests and experiences would seem to be fundamental teaching precepts.

The second major implication for teachers that arose from this study was the lack of enjoyment and understanding expressed by the students. This state of affairs could have resulted from learning experiences that did not attend to the different probability types, inappropriate processing behaviours and/or lack of worthwhile concrete experiences. Therefore, whilst providing students with both a sense of purpose and relevant problem situations may help to engender enjoyment as well as understanding of probability, there are other aspects which need to be considered.

One aspect is that teachers need to be aware of the different types of probability so that instructional experiences are planned to meet the desired learning outcomes. For example, Ali, probably had had more concrete experiences than most of the students but she also appeared to be the most confused, even when taking into account her apparent impoverished common fraction knowledge. Ali was unable to distinguish between formal and frequentist types of probability (Hawkins & Kapadia, 1984) and the result was frustration for Ali as indicated by her protocol when she blurted out: *See, that's what I hate about probability!* This outburst was occasioned by the task that was related to frequentist probability (see Baturo, 1992, Task 2).

Ali's behaviour is not an isolated case because the literature has documented the disastrous consequences that may occur if instruction includes probability experiments that test frequentist notions (e.g., validating, through repeated trials, theoretical calculations or subjective hypotheses) in the vague hope of making probability more meaningful and enjoyable. For example:

> In general, the results show a disparity between the quality of predictions and the justifications given for them. The vast majority of students gave mathematically sound predictions. However, when they found that these predictions were not confirmed by the data, many pupils reverted to past experience, hunches or cynicism . . . Pupils become confused, either clinging to what they believed must be the mathematically correct answer despite contrary results, or abandoning theory altogether.
>
> (Hawkins & Kapadia, 1984, p. 360

Apart from the confusion that may result from not distinguishing between the probability types, anger or frustration may also be provoked unwittingly through materials that stimulate inappropriate processing behaviours. For example, to interpret the spinner used in Task 1B, many students resorted to perceptual processing. This behaviour supports Lovett and Singer's (1991) finding that many children have a non-quantitative understanding of probability, preferring to use perceptual strategies when both perceptual and quantitative strategies are supported. Lovett and Singer's finding related to young children (kindergarteners to Grade 5) so the Year 12 students' use of perceptual processing in this task could be interpreted as atavistic behaviour. The protocols of the students who appeared to be in the process of reconciling the perceptual and cognitive processes required for the correct interpretation of the spinner revealed the frustration that arises from situations for which they are not quite ready in an intellectual sense.

The third major finding of the questionnaire related to the students' lack of concrete experiences and its effect on both their enjoyment and understanding of probability. This finding is not surprising because one of the maxims of modern pedagogical philosophy is that students enjoy and understand mathematics when they are actively involved in the learning process (Department of Education, Queensland, 1987b). In this study, some students commented that their study of probability would have been less boring had they physically carried out the textbook activities, thus supporting the importance of concrete experiences that has been documented in the literature.

Concrete experiences are vital because they not only foster mental representation of the particular concept to be taught but also mental representations of the processes involved. That is, concrete experience, as opposed to pictorial experience, can provide a kinaesthetic as well as a visual memory of the concept and process being taught, thus providing an enriched base for computational knowledge. Concrete experiences and concomitant discussion also help to make sense of the specific mathematical terminology.

However, the concrete experiences should be such that the desired mathematical principle is abstracted (Dienes, 1969). In promoting elementary probability notions (which appear to be closely connected to a sound knowledge of fractions), the incorporation of spinners in early concrete probability experiences is vital as spinners belong to the area models that were discussed earlier. Perhaps the most important implication for the teaching of the elementary notions of probability emanated from the analysis of the students' results in all stages of this study. It should be remembered, however, that this study focused on what it means for Year 12 students to know the elementary probability notions because, by this stage, of their schooling, it was believed that they should be more expert than novice in their understanding of any piece of curriculum. Yet, in many cases, a good performance did not reflect expert knowledge. In fact, the expert knowledge, when probed, was often quite fragile. This phenomenon may be the result of instruction that does not take into account the different knowledge types and the connections that apparently should be made between the types.

Those students who had good principled conceptual probability knowledge also had a very sound principled conceptual fraction knowledge, in general, and very good principled conceptual common fraction knowledge in particular (Baturo, 1992). As Fischbein (1975, p. 176) points out, "The mathematical concept of probability takes the form W/(W+B)(the W and B referred to the white and black marbles which were used in the tasks he administered to the students in his study.] Therefore, it is assumed that an improvement in common fraction knowledge will have a concomitant improvement on probability knowledge.

Using common fractions to represent probabilistic situations is far more descriptive of the sample space than per cents or decimal fractions. This is because, with common fractions, the numerator represents the number of favourable outcomes and the denominator represents the total number of equally possible outcomes. Therefore, both components to be considered are visible. For example, of the following symbolic notations which are used to represent probabilistic situations, [$P_{(red)} = {}^{3}/_{8}$, $P_{(red)} = 37.5\%$, $P_{(red)} = 37.5$], the common fraction is more likely to promote a clear and simple mental representation of the probabilistic situation. Therefore, when children are introduced to the formal probability notation, it seems essential that the probabilities be in common fraction form rather than in per cent or decimal fraction form.

Furthermore, there appears to be more value in not reducing the terms of a common fraction when representing probabilistic situations in notational form. For example, to represent a set of 8 marbles, 6 of which are red, $P_{(red)} = 6/8$ is a more accurate representation of the problem than $P_{(red)} = 3/4$.

Finally, the role of language in helping students formulate the specific knowledge required within each knowledge and in helping students connect one type of knowledge to another cannot be overestimated. In this study, appropriate probability language appeared to be inaccessible to many students.

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